

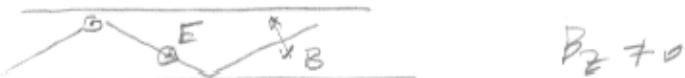
## Metal Waveguides

including vector character and boundary cond.  
vectors; anticipate orientation of fields rel. to walls.  
work solutions

TEM  $\vec{E} \perp \vec{B} \perp \hat{z}$        $E_z = B_z = 0$

example: free space plane wave.

TE  $\vec{E}$  is transverse only       $\vec{E}_z = 0$



TM  $B_z = 0$

"transverse" means field is  $\perp \hat{z}$

Remember, once modes are known (eigenfunctions)  
an arbitrary solution can be composed

$$\vec{E} = \sum_m E_{TEm} + \sum_n E_{Tmn}$$

modes are a complete set that can describe  
any function compatible with boundary conditions.

Strip lines: plane parallel conductors

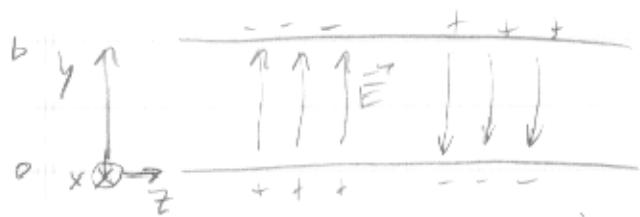
assume perfect conductor  $\rightarrow$  surface charge, current.

TEM is allowed:

$$\frac{E}{\vec{B}} \longrightarrow \vec{B}$$

same velocity as in vacuum ( $c_0/\epsilon_0$ )

no modes,



For TEM  $\vec{E} = \hat{y} E_0 e^{i(kz - \omega t)}$  along  $\hat{x}$   $E \parallel$  to metal is zero.

since  $\nabla \cdot \vec{E} = 0$  inside wg,

$$\rightarrow \frac{\partial E}{\partial y} = 0 \quad \vec{E} \text{ is constant}$$

finite value of  $E_z$  at metal  $\rightarrow$  surf. charge.

- plates are not connected electrically.

get  $\vec{B}$  from  $\vec{E}$ :

$$\nabla \times \vec{E} = \frac{i\omega}{c} \vec{B} \quad E \text{ varies only w/ } z$$

$$-ikE_0 \hat{x} = \frac{i\omega}{c} \vec{B}$$

$$B = -\hat{x} E_0 \quad \omega = kc$$

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} E_y$$

from wave eqn.

Wave behaves similar to free wave, but  $\vec{E}$  only along  $y$ ,  
 $\vec{E}$  is confined b/w. plates.

in practice: striplines



- no cutoff i.e. prop. long  $\lambda$  in small guide.

coax. cable:



wave velocity determined by dielectric b/w. conductors.

## Guided wave solutions

Wave equation:

Assume: no variation in structure with  $z$

$n(x, y) = \sqrt{\epsilon\mu}$  is piecewise constant, isotropic

then within each region,

$$\nabla \cdot \vec{D} = \nabla \cdot (\epsilon \vec{E}) = \epsilon \nabla \cdot \vec{E} = 0 \quad \text{no free charges}$$

∴ use wave eqn in each region

$$\nabla^2 \vec{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{where } \epsilon, \mu \text{ depend on region}$$

satisfy the boundary conditions to connect the solutions.

separation of variables

$$\vec{E}(\vec{r}, t) = \vec{E}_T(\vec{r}_T) e^{i(k_z z - \omega t)} \quad \begin{array}{l} \text{wave prop to right} \\ \text{if } k_z > 0 \end{array}$$

All fields vary with  $e^{-i\omega t}$ .

Since eqn is linear, we can later add different solutions with diff't  $\omega$ 's (e.g. Fourier series)

We can write Laplacian as

$$\nabla^2 \rightarrow \nabla_T^2 + \frac{\partial^2}{\partial z^2}$$

$$\text{in Cartesian, } \nabla_T^2 f = (\partial_x^2 + \partial_y^2) f$$

$$\text{in cylindrical, } \nabla_T^2 f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_\phi^2 f$$

Now wave eqn. is

$$\nabla_T^2 \vec{E}_T(\vec{r}_T) - k_z^2 \vec{E}_T + \frac{n^2 \omega^2}{c^2} \vec{E}_T = 0$$

rearrange:

$$\nabla_T^2 \vec{E}_T + n^2(\vec{r}_T) k_0^2 \vec{E}_T = k_z^2 \vec{E}_T$$

w/ similar eqn for  $\vec{B}_T$

In Cartesian coord, allowed solutions are of form  $e^{ik_x x} e^{ik_y y}$

$$\rightarrow k_x^2 + k_y^2 + k_z^2 = n^2 k_0^2 \text{ in each region}$$

no assumptions here about  $k$ 's and  $n$ 's being real

Also, note that  $k_z$  is a constant through the system

- each value of  $n \rightarrow$  different values of  $E_T(\vec{r}_T)$ ,  $k_x, k_y$
- all fields have  $e^{ik_z z}$  dependence



this is phase continuity = Snell's law

Structure of wave equation is an eigenvalue eqn:

$$[\nabla_T^2 + n^2(\vec{r}_T) k_0^2] \vec{E}_T = k_z^2 \vec{E}_T$$

same structure as Schrödinger wave eqn.

where  $\hat{H} \Psi = E \Psi$

$$\text{and } \hat{H} = \frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) = -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r})$$

boundary conditions can impose quantization

$\rightarrow$  discrete eigenvalues  $k_z$

revisit plane  $\parallel$  conductors; TE  
 $\vec{E}_T = \hat{x} (a e^{ik_y y} + b e^{-ik_y y})$

$$E = 0 \text{ at } y = 0 \rightarrow a = -b \quad \therefore \text{use } \sin k_y y$$

$$\vec{E}_T = \hat{x} E_0 \sin(k_y y)$$

$$E = 0 \text{ at } y = b \rightarrow k_y b = m\pi$$

$$\text{now } k_z = \sqrt{n^2 k_0^2 - k_y^2} \quad k_y = \frac{m\pi}{b}$$

TM:

$$B_z = 0 \quad E_y, E_z \neq 0 \quad \nabla \cdot \vec{B} = \partial_x B_x + \partial_y B_y = 0$$

$B_y = 0$  or constant, but at conductor  $B_y \rightarrow 0$

$\therefore$  only have  $B_x, E_y, E_z$

choose which to solve for.

redundancy in Maxwell eqns

$$\nabla \cdot \vec{E} = \partial_y E_y + \partial_z E_z = 0 \quad \textcircled{1}$$

$$(\nabla \times \vec{E})_x = i k_0 B_x \rightarrow \partial_y E_z - i k_z E_y = i k_0 B_x \quad \textcircled{2}$$

$$\nabla \times \vec{B} = -i k_0 \vec{E} \rightarrow i k_z B_x = -i k_0 E_y \quad \textcircled{3}$$

$$-\partial_y B_x = -i k_0 E_z \quad \textcircled{4}$$

Any will work, but if we solve for  $E_z$ ,  
 can get others thru derivatives

$$\textcircled{3} \quad E_y = -\frac{k_z}{k_0} B_x$$

$$\rightarrow \textcircled{4} \quad \partial_y E_z + i \frac{k_z^2}{k_0} B_x = i k_0 B_x$$

$$k_0 \partial_y E_z = i(k_0^2 - k_z^2) B_x = i k_y^2 B_x$$

$$B_x = -i \frac{k_0}{k_y^2} \partial_y E_z$$

$$E_y = -\frac{k_z}{k_0} B_x = i \frac{k_z}{k_y^2} \partial_y E_z$$

message: can get all components for TM directly from  $E_z$

for TE, can get all from  $B_z$

This is especially valuable in 2D waveguides, since  
B.C. are uniform at all walls.

i. for TM solve for  $E_z$  with  $E_z \rightarrow 0$  at walls.

for TE solve for  $B_z$

$$\text{from } \nabla \times \vec{B} = -ik_0 \vec{E}$$

$$\frac{\partial B_z}{\partial y} - ik_z B_y = ik_0 E_x$$

$$\text{at walls } B_y = E_x = 0 \quad \therefore \frac{\partial B_z}{\partial y} = 0$$

derivative of  $B_z \rightarrow 0$  at walls TE

HM Book has all connections btw fields derived

just remember in 2D waveguides to solve for  $E_z$  component

## Rectangular metal waveguides

solve for longitudinal field

TE: solve for  $B_z$  with  $\partial B_z / \partial n = 0$   
i.e. zero slope at walls

TM: solve for  $E_z$  with  $E_z = 0$  at walls.

extend 1-D solutions -

remember separation of variables  $\rightarrow$  product of solns

$$TE: B_z(x, y, z) = B_{z0} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{i(k_z^{(n,m)} z - \omega t)}$$

modes identified by index

TEM<sub>mn</sub>

$$\begin{aligned} \text{propagation constant } k_z^{(m,n)} &= k_z \\ k_z &= \sqrt{k_0^2 - k_x^2 - k_y^2} \\ &= \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \end{aligned}$$

if  $a > b$ , lowest mode (largest  $k_z$ ) is TE<sub>10</sub>

(TE<sub>00</sub> = TEM<sub>00</sub> can't propagate)

> cutoff for any mode is where  $k_z = 0$

as freq. is tuned, reach diff'nt cutoffs;

$$\frac{w_{mn}^2}{c^2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \rightarrow w_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

> single-mode: if  $w_{10} < w < w_{01}$

this range is single-mode bandwidth.