8. The thermal equilibrium values of the physical properties of a system are defined as averages over all states accessible when the system is in contact with a large system or reservoir. If the first system also is large, the thermal equilibrium properties are given accurately by consideration of the states in the most probable configuration alone.

PROBLEMS

1. Entropy and temperature. Suppose $g(U) = CU^{3N/2}$, where C is a constant and N is the number of particles. (a) Show that $U = \frac{3}{2}N\tau$. (b) Show that $(\partial^2\sigma/\partial U^2)_N$ is negative. This form of $g(U)$ actually applies to an ideal gas.

2. Paramagnetism. Find the equilibrium value at temperature τ of the fractional magnetization

$$M/Nm = 2\langle s\rangle/N$$

of the system of N spins each of magnetic moment m in a magnetic field B. The spin excess is $2s$. Take the entropy as the logarthithm of the multiplicity $g(N,s)$ as given in (1.35):

$$\sigma(s) \simeq \log g(N,0) - 2s^2/N \ , \tag{40}$$

for $|s| \ll N$. *Hint*: Show that in this approximation

$$\sigma(U) = \sigma_0 - U^2/2m^2B^2N \ , \tag{41}$$

with $\sigma_0 = \log g(N,0)$. Further, show that $1/\tau = -U/m^2B^2N$, where U denotes $\langle U\rangle$, the thermal average energy.

3. Quantum harmonic oscillator. (a) Find the entropy of a set of N oscillators of frequency ω as a function of the total quantum number n. Use the multiplicity function (1.55) and make the Stirling approximation $\log N! \simeq N\log N - N$. Replace $N - 1$ by N. (b) Let U denote the total energy $n\hbar\omega$ of the oscillators. Express the entropy as $\sigma(U,N)$. Show that the total energy at temperature τ is

$$U = \frac{N\hbar\omega}{\exp(\hbar\omega/\tau) - 1}. \tag{42}$$

This is the Planck result; it is derived again in Chapter 4 by a powerful method that does not require us to find the multiplicity function.

4. ***The meaning of "never."*** It has been said* that "six monkeys, set to strum unintelligently on typewriters for millions of years, would be bound in time to write all the books in the British Museum." This statement is nonsense, for it gives a misleading conclusion about very, very large numbers. Could all the monkeys in the world have typed out a single specified book in the age of the universe?†

Suppose that 10^{10} monkeys have been seated at typewriters throughout the age of the universe, 10^{18} s. This number of monkeys is about three times greater than the present human population‡ of the earth. We suppose that a monkey can hit 10 typewriter keys per second. A typewriter may have 44 keys; we accept lowercase letters in place of capital letters. Assuming that Shakespeare's *Hamlet* has 10^5 characters, will the monkeys hit upon *Hamlet*?

(a) Show that the probability that any given sequence of 10^5 characters typed at random will come out in the correct sequence (the sequence of *Hamlet*) is of the order of

$$\left(\tfrac{1}{44}\right)^{100\,000} = 10^{-164\,345} ,$$

where we have used $\log_{10} 44 = 1.64345$.

(b) Show that the probability that a *monkey-Hamlet* will be typed in the age of the universe is approximately $10^{-164\,316}$. The probability of *Hamlet* is therefore zero in any operational sense of an event, so that the original statement at the beginning of this problem is nonsense: one book, much less a library, will never occur in the total literary production of the monkeys.

5. ***Additivity of entropy for two spin systems.*** Given two systems of $N_1 \simeq N_2 = 10^{22}$ spins with multiplicity functions $g_1(N_1,s_1)$ and $g_2(N_2,s - s_1)$, the product g_1g_2 as a function of s_1 is relatively sharply peaked at $s_1 = \hat{s}_1$. For $s_1 = \hat{s}_1 + 10^{12}$, the product g_1g_2 is reduced by 10^{-174} from its peak value. Use the Gaussian approximation to the multiplicity function; the form (17) may be useful.

(a) Compute $g_1g_2/(g_1g_2)_{\max}$ for $s_1 = \hat{s}_1 + 10^{11}$ and $s = 0$.

(b) For $s = 10^{20}$, by what factor must you multiply $(g_1g_2)_{\max}$ to make it equal to $\sum_{s_1} g_1(N_1,s_1)g_2(N_2,s - s_1)$; give the factor to the nearest order of magnitude.

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* J. Jeans, *Mysterious universe*, Cambridge University Press, 1930, p. 4. The statement is attributed to Huxley.

† For a related mathematico-literary study, see "The Library of Babel," by the fascinating Argentine writer Jorge Luis Borges, in *Ficciones*, Grove Press, Evergreen paperback, 1962, pp. 79–88.

‡ For every person now alive, some thirty persons have once lived. This figure is quoted by A. C. Clarke in *2001*. We are grateful to the Population Reference Bureau and to Dr. Roger Revelle for explanations of the evidence. The cumulative number of man-seconds is 2×10^{20}, if we take the average lifetime as 2×10^9 s and the number of lives as 1×10^{11}. The cumulative number of man-seconds is much less than the number of monkey-seconds (10^{28}) taken in the problem.

(c) How large is the fractional error in the entropy when you ignore this factor?

6. Integrated deviation. For the example that gave the result (17), calculate approximately the probability that the fractional deviation from equilibrium δ/N_1 is 10^{-10} or larger. Take $N_1 = N_2 = 10^{22}$. You will find it convenient to use an asymptotic expansion for the complementary error function. When $x \gg 1$,

$$2x\exp(x^2)\int_x^\infty \exp(-t^2)\,dt \approx 1 + \text{small terms}.$$