11_30_07

Note Title 11/15/2006

$$\nabla^2 \psi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

Laplaces equation in spherical coordinates

Make our normal
Sep. of variables guess

$$\psi(r,\theta,\phi) = R(r)\,P(\theta)\,Q(\phi)$$

So, e.g. $\frac{\partial \psi}{\partial r} = R'(r)\,P(\theta)\,Q(\phi)$

$$\nabla^2 \psi = \frac{1}{r^2} P \Phi \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$$

$$+ \frac{R\Phi}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)$$

$$\frac{RP}{r^2 \sin^2\theta} \frac{d^2\Phi}{d\varphi^2} = 0$$

Always now divide by $RP\Phi$

$$\nabla^2 \psi = \frac{1}{r^2 R} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$$

$$+ \frac{1}{r^2 \sin\theta \, P} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta \, \Phi} \frac{d^2\Phi}{d\varphi^2} = 0$$

we can isolate the $\Phi(\phi)$ part if we multiply by $r^2 \sin^2\theta \Rightarrow$

$$\nabla^2 \psi = \frac{\sin^2\theta}{R} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{\sin\theta}{P} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)$$

$$= -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \equiv m^2$$

$$\boxed{\Phi'' + m^2 \Phi = 0}$$ first ODE

so $\Phi(\phi) \propto e^{im\phi}$

Does m have to be an integer?

Suppose $m = .1$

$Q(0) = 1$

$Q(2\pi) = e^{i \cdot .1 \cdot 2\pi} \neq 1$

Like a Boundary condition the periodicity forces m to be an integer

Back to r, θ:

$$\frac{\sin^2\theta}{R} \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)$$

$$+ \frac{\sin\theta}{P} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) = m^2$$

Divide by $\sin^2\theta$ and move θ part to RHS

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$$

$$= -\frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{m^2}{\sin^2\theta}$$

r on the left θ on right

r is easiest to do first

$$\boxed{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = k^2}$$

Second ODE

Guess a trial Solution

$$R(r) = A r^{\alpha}$$

$$R' = A\alpha r^{\alpha-1}$$

$$r^2 R' = A\alpha r^{\alpha+1}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) =$$

$$\frac{1}{\cancel{A}\,\cancel{r^{\alpha}}}\cdot \cancel{A}\,(\alpha+1)\alpha\,\cancel{r}^{\alpha} = k^2$$

$$\Rightarrow \quad \alpha(\alpha+1) = k^2$$

quadratic for unknown α

we solve this directly for α, but solutions are cumbersome

However:

suppose $k^2 = \ell(\ell+1)$

then quadratic becomes

$$\alpha(\alpha+1) = \ell(\ell+1)$$

factor

$$\Leftrightarrow \quad (\alpha-\ell)(\alpha+(\ell+1)) = 0$$

so either $\boxed{\alpha = \ell \quad \text{or} \quad \alpha = -(\ell+1)}$

recap: we can get solutions to the radial part of Laplace's eqn if we write the separation constant as

$$k^2 = \ell(\ell+1)$$

then $R(r) = A r^{\alpha}$ works for $\alpha = \ell$, $\alpha = -(\ell+1)$

$$\boxed{R(r) = A_\ell r^{\ell} + B_\ell r^{-(\ell+1)}}$$

solution to second ODE

Finally we get to the θ part

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P(\theta)=0}$$

3rd ODE : Legendre's Equation

Exercise Let $x=\cos\theta$

Show that becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P(x)$$

Solutions of this equation involve 2 indices ℓ, m.

Lets call them

$$P_{\ell,m}(x)$$

Then $\psi(r,\theta,\varphi) = R(r)P(\theta)\Phi(\varphi)$

$$= \begin{cases} r^{\ell} \; P_{\ell m}(\cos\theta)\, e^{im\varphi} \\ r^{-(\ell+1)} \; P_{\ell m}(\cos\theta)\, e^{im\varphi} \end{cases}$$

$$\underbrace{P_{\ell m}(\cos\theta)}_{\text{Legendre polynomials}} e^{im\varphi} \equiv \underbrace{Y_{\ell m}(\theta,\varphi)}_{\text{spherical Harmonics}}$$

NB: we have not yet actually solved Legendre's equation

Major Result!

Any solution of $\nabla^2 \Phi = 0$ can be written as

$$\boxed{\Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^{l} + B_{lm} r^{-(l+1)} \right) Y_{lm}}$$

we have not yet proved that m runs from $-l, l$ but enough for today.

$$Y_{00} = \sqrt{\frac{1}{4\pi}}$$

$$Y_{10} = \sqrt{3/4\pi} \quad \cos\theta$$

$$Y_{1\,\pm1} = \sqrt{\frac{3}{8\pi}} \quad \sin\theta \; e^{\pm i\varphi}$$

$\vdots$

See my lecture notes on "sep of variables & special functions" wiki week of 11/26.

Also BOAS ch. 13 sec. 7

$$\psi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^{l} + B_{lm} r^{-(l+1)}\right) Y_{lm}$$

spherical symm. e.g. point charge

$\Rightarrow \quad l=0 \quad m=0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$

$$\psi(r) = \left(A_{00} r^{0} + B_{00} r^{-1}\right) Y_{00}$$

$$\psi(r) = 1/r$$

important special case
$m=0 \Rightarrow$ No ϕ dependence

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_{\ell m}(x)$$

Associated Legendre functions

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P_\ell(x)$$

ordinary Legendre functions.

Turns out:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

$$\underline{P_{\ell m}(x) = (-1)^m (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)}$$

$P_0(x) = 1$

$P_1(x) = x$ [i.e. $P_1(\cos\theta) = \cos\theta$]

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

since $x = \cos\theta$ $\quad x \in [-1, 1]$

Any function on $[-1, 1]$ can be written

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x)$$

$$\left(P_l(x), P_m(x)\right) = \frac{2}{2l+1} \delta_{lm} \Rightarrow$$

$$A_m = \frac{2m+1}{2} \left(P_m, f(x)\right)$$

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suppose you wanted to create your own set of orthogonal polynomials on $[-1, 1]$

$Q_0 = a_0$ $\quad$ start with a const.

$Q_1(x) = a_1 + b_1 x$

$Q_2(x) = a_2 + b_2 x + c_2 x^2$

$\vdots$

Require that $(Q_i, Q_j) = \delta_{ij}$

This means that your functions Q_i are normalized to 1 and mutually orthogonal

So: $(Q_0, Q_0) = \int_{-1}^{1} a_0^2 \, dx = a_0^2 x \Big|_{-1}^{1}$

$= 2a_0^2 = 1$

normalization

$\Rightarrow a_0^2 = \frac{1}{2}$ $\boxed{a_0 = \frac{1}{\sqrt{2}}}$

$\boxed{Q_0(x) = \frac{1}{\sqrt{2}}}$

$Q_1(x) = a_1 + b_1 x$ $(Q_1, Q_1) = 1$

$(Q_1, Q_0) = 0$

$(Q_0, Q_1) = \int_{-1}^{1} \frac{1}{\sqrt{2}} \left[a_1 + b_1 x\right] dx = 0$

$\frac{1}{\sqrt{2}} \left[2a_1 + \underbrace{b_1 \frac{x^2}{2} \Big|_{-1}^{1}}_{0} \right] = 0$

$\Rightarrow \boxed{a_1 = 0}$

$$(Q_1, Q_1) = 1 = \int_{-1}^{1} b_1^2 x^2$$

$$\Rightarrow \quad \frac{2}{3} b^2 = 1 \quad \Rightarrow \boxed{b = \sqrt{\frac{3}{2}}}$$

$$Q_1(x) = \sqrt{\frac{3}{2}}\, x$$

$$Q_2(x) = a_2 + b_2 x + c_2 x^2$$

$$\int_{-1}^{1} Q_0(x)\, Q_2(x)\, dx = \sqrt{2}\left[a_2 + c_2/3\right] = 0$$

$$\int_{-1}^{1} Q_1(x)\, Q_2(x)\, dx = \sqrt{\frac{2}{3}}\, b_2 = 0$$

$$\boxed{b_2 = 0} \qquad a_2 = -\frac{1}{3} c_2$$

finally $\displaystyle\int_{-1}^{1} Q_2(x)\, Q_2(x)\, dx = \frac{8 c_2^2}{45}$

$$\Rightarrow \boxed{c_2 = \sqrt{\frac{45}{8}} = \frac{3}{2}\sqrt{5/2}} \qquad \boxed{a_2 = -\frac{1}{3}\sqrt{45/8} = -\frac{1}{2}\sqrt{5/2}}$$

$$Q_2(x) = -\frac{1}{2}\sqrt{5/2} + \frac{3}{2}\sqrt{5/2}\, x^2$$

$$\boxed{\begin{aligned} Q_2(x) &= \tfrac{1}{2}\sqrt{\tfrac{5}{2}}\left[3x^2 - 1\right] \\ Q_1(x) &= \sqrt{3/2}\; x \\ Q_0(x) &= \tfrac{1}{\sqrt{2}} \end{aligned}}$$

whereas the Legendre polynomials are

$$\left.\begin{aligned} P_3(x) &= \tfrac{1}{2}(3x^2 - 1) \\ P_2(x) &= x \\ P_1(x) &= 1 \end{aligned}\right\}$$ But these are not normalized to $\underline{1}$

$$(P_\ell(x), P_m(x)) = \frac{2}{2\ell+1}\,\delta_{\ell m}$$

normalization factors

so $(P_0, P_0) = 2 \Rightarrow \frac{1}{\sqrt{2}}$

$(P_1, P_1) = \frac{2}{3} \qquad \sqrt{3/2}$

$(P_2, P_2) = \frac{2}{5} \qquad \sqrt{5/2}$

Our Q functions are normalized Legendre polynomials