Homework 4

constants and unit definitions

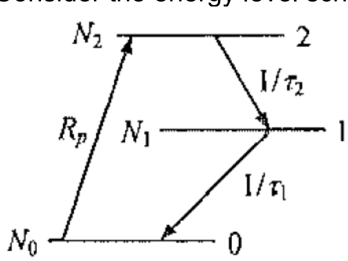
This is a block of code I use to define my constants. The internal units for calculations are SI, but the numbers can be entered in other units.

```
s = 1; ms = 10^-3 s; us = 10^-6 s; ns = 10^-9 s; ps = 10^-12 s;  fs = 10^-15 s;
Hz = 1; GHz = 10^9 Hz; THz = 10^12 Hz;
m = 1; cm = 10^-2 m; mm = 10^-3 m; um = 10^-6 m; nm = 10^-9 m; angs = 10^-10 m;
kg = 1; g = 10^-3;
J = 1; mJ = 10^-3 J; eV = e J; keV = 10^3 eV; MeV = 10^6 eV;
W = 1; kW = 10^3 W; GW = 10^9 W; mW = 10^-3 W;
Torr = 1 / 760;
K = 1;

c = 299 792 458 m / s;
eps0 = 8.854187817 × 10^-12;  (*permittivity of free space*)
re = 2.81794092 × 10^-15 m;
e = 1.60217733 × 10^-19;          (*Coul electron charge*)
Natm = 760 × 3.3 × 10^16 / cm^3; (*no.particles in 1 atm*)
me = 9.1093897 × 10^-31 kg;
mp = me / 0.000544617013;
hbar = 1.05457266 × 10^-34 J s;
h = 2 π hbar;
kB = 1.3806503 × 10^-23 J / K;
```

Problem 1
rate equations for a 3 level laser

Consider the energy level scheme for a 3-level laser shown in the figure below.



In a conventional 3-level system such as ruby, the inversion is on the 1 to 0 transition and most of the population must be transferred out of level 0 to get a population inversion. In this case, we will be looking for an inversion on the 2 to 1 transition.
Starting at t = 0, where the atoms are in the ground state, the atoms are raised from level 0 to level 2 at a pump rate R_p. The lifetime of levels 1 and 2 are τ_1 and τ_2 respectively.

a) Write the rate equations for the population densities, N_1 and N_2, of level 1 and 2 respectively. Assume that the ground state 0 is not depleted to any significant

extent and neglect stimulated emission from 2 to 0.

b) Find the steady state solution. What condition is required for a population inversion on the 2 to 1 transition?

c) Analytically calculate $N_1(t)$ and $N_2(t)$ for the initial conditions $N_1(0) = 0$ and $N_2(0) = 0$. You can use DSolve[ ] in *Mathematica* to check your work, but show the steps how to get the solution. *Hint:* the differential equation for the population of level 1 can be solved by multiplying both sides by the factor $\exp(t/\tau_1)$. In this way the left-hand side of the preceding differential equation becomes a perfect differential.

d) Use Mathematica to plot the population densities ($N_1(t)$ and $N_2(t)$) and the population inversion ($N_2(t) - N_1(t)$) on the same plot. Make two plots using the following two input values (you may pick an arbitrary value of R_p):

I. $\tau_1 = 2\ \mu s$, $\tau_2 = 1\ \mu s$

II. $\tau_1 = 1\ \mu s$, $\tau_2 = 2\ \mu s$

Is it possible to achieve a population inversion for each of the two cases? If so, what are the requirements on the pump pulse?

Problem 2
Generalized gain saturation

A more general derivation of the saturation in a two level system. In this version, we have pump rates (/vol/time) R1 and R2, and overall lifetimes out of the levels $\tau 1$ and $\tau 2$. We also include the effect of degeneracies on levels 1 and 2, g1 and g2. We'll work with the beam intensity and the cross-section rather than the pump rate $W = \frac{I \sigma_{21}}{h \nu_L}$. With these included, the rate equations are

$$\frac{d N_1}{d t} = R_1 + \Delta N^* \sigma_{21} \frac{I}{h \nu_L} - \frac{N_1}{\tau_1} + N_2 A_{21}$$

$$\frac{d N_2}{d t} = R_2 - \Delta N^* \sigma_{21} \frac{I}{h \nu_L} - \frac{N_2}{\tau_2}$$

where

$$\Delta N^* = N_2 - \frac{g_2}{g_1} N_1 = N_2 - r_g N_1$$

is the population inversion density. To make the notation more compact, I am writing $r_g = g_2/g_1$. Remember that in this formulation, spontaneous emission out of level 2 is included in the lifetime in that level, τ_2.

a. First set the time derivatives = 0 for steady state, then calculate the steady state expressions for N_1^{ss} and N_2^{ss}. Then calculate the steady-state value of ΔN^*, to show that

$$\Delta N^*(I) = N_2^{ss} - \frac{g_2}{g_1} N_1^{ss} = \frac{R_2 \tau_2 [1 - r_g A_{21} \tau_1] - r_g R_1 \tau_1}{1 + \sigma_{21} I \frac{1}{h \nu_L} [\tau_2 + r_g \tau_1 - r_g A_{21} \tau_1 \tau_2]}$$

b. This equation can be written in the standard form

$$\Delta N^*(I) = \frac{\Delta N^*(0)}{1 + I/I_s}$$, with the saturation intensity $I_s = \frac{h \nu_L}{\sigma_{21} \tau_R}$.

Write expressions for $\Delta N^*(0)$ and the “recovery time” τ_R and give a description of what each of those terms mean physically.

Problem 3: wave interference

a. Write an expression for a plane wave traveling in a direction θ_x from the z-axis so that the k-vector is in the x-z plane and the wave is polarized along the y direction. Use complex (phasor) notation following the $e^{-i\omega t}$ convention.

b. The wave is incident on a mirror at z = L placed so that its surface normal is in the $-\hat{z}$ direction, so the component of the total E-field along the surface must be zero. Write an expression for the reflected wave and find its amplitude and phase so that the total E-field satisfies the boundary conditons.

c. Write an expression for the sum of the fields, and calculate the time averaged intensity of the combined field. Since we anticipate a standing wave in the z-direction, simplify the expression to show this explicitly (i.e. in terms of a sine or cosine function). The overall expression for the field will still be complex.