

## Response functions in frequency and time domains

We've been pretty loose about something so far. Usually, when we write fields, we write them in the time domain, as such:

$$\vec{E}(\vec{x}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

And there are various relationships that we use that involve  $\vec{E}$ , such as:

$$\vec{P} = \epsilon_0 \chi \vec{E}, \quad \vec{D} = \epsilon \vec{E}$$

But these aren't time domain relationships. As written, they're frequency domain:

$$\vec{P} = \epsilon_0 \chi(\omega) \vec{E}(\omega), \quad \vec{D}(\omega) = \epsilon \vec{E}(\omega)$$

They tell you what kind of polarization you get given a driving field of frequency  $\omega$ . They're not in general correct time domain expressions - materials don't polarize instantaneously.

The amount by which a material is polarized at some specific time  $t$  depends on the whole history of the material up to time  $t$ . A proper time domain representation of the relationship between  $\vec{P}$  and  $\vec{E}$  would be:

$$\vec{P}(t) = \epsilon_0 \int_{-\infty}^t \chi(t-t') \vec{E}(t') dt' \quad (1)$$

The susceptibility  $\chi$  is what we call a response function (linear, as written). It tells us how a system responds to some external influence. The fact that the integral cuts off at  $t'=t$  guarantees causal behavior: Effects will necessarily follow causes.

The Fourier transform of (1) ends up being

$$\vec{P}(\omega) = \epsilon_0 \tilde{\chi}(\omega) \vec{E}(\omega)$$

which is what we're familiar with.

But that means writing something like:

$$\vec{D}(\vec{x}, t) = \epsilon_0 \chi \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

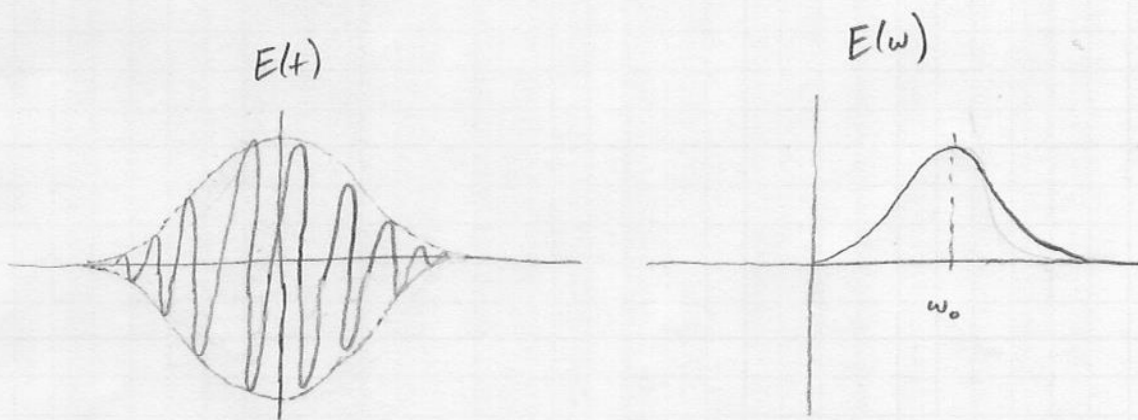
isn't really correct. You can get away with not paying attention to the difference between time and frequency domain expressions in statics (not surprisingly), and even in dynamics if you're limited to harmonic driving functions that have been going for a while and susceptibilities that are the same for all  $\omega$ , but eventually you're going to get in trouble.

## Causality and Kramers-Kronig

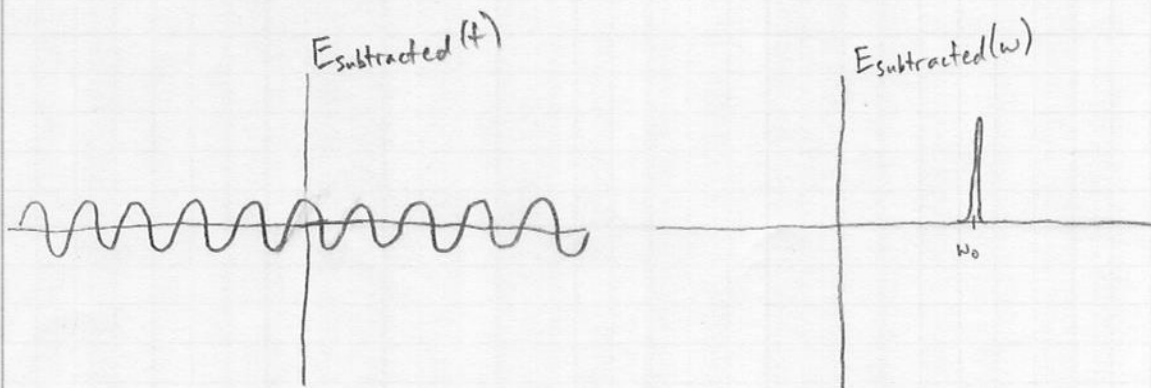
Let's build on that notion of causality a bit. First, we'll take stock of some of what we've learned:

- Real materials are usually dispersive, which is to say  $n$  is a function of frequency,  $n(\omega)$
- Real materials are usually absorptive, which we can model with a complex  $k$  or  $n$ , like  $n = n_1 + i n_2$
- Real light (especially in pulse form) is composed of many frequencies all mixed together

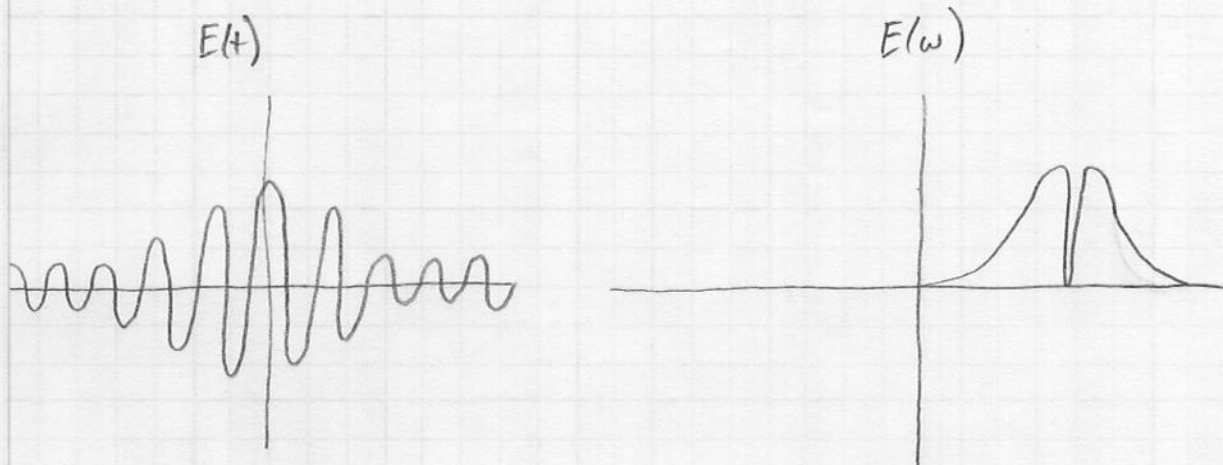
If we combine all this in a particular way, something very interesting happens. Suppose we have a Gaussian pulse of light, with its corresponding time and frequency domain representations



Now suppose this pulse passes through some material that's very absorbent right at the carrier frequency  $\omega_0$ . We subtract out a spike at that frequency, in both domains:



Which leaves us with fields like:



There's something very bad about that time domain result. It has a sine wave going off towards  $t = \pm\infty$ . That means that because of an absorption event that happened now, the pulse developed a tail yesterday. That's not supposed to happen in a causal theory.

In reality, this doesn't happen because the absorption isn't the only event. All the other frequencies get phase shifted to varying degrees in such a way that the new spectrum still amounts to a pulse that doesn't reach back in time.

This turns out to be pretty important. To have a causal theory of E+M, absorption has to be accompanied by a bunch of phase shifts that almost certainly are different at different frequencies. And so since no material is perfectly absorption free, no real material can be dispersionless!

Absorption  $\Rightarrow$  Dispersion

Not surprisingly, this result is encoded in a mathematical theorem. In the time domain, polarization and field are related by a response function, the susceptibility  $\chi$ :

$$\vec{P}(t) = \epsilon_0 \int_{-\infty}^t \chi(t-t') \vec{E}(t') dt'$$



And there's a corresponding frequency domain relationship:

$$\tilde{D}(\omega) = \epsilon_0 \tilde{\chi}(\omega) \tilde{E}(\omega)$$

Now,  $\tilde{\chi}$  is what eventually gets bundled up into  $n$  by way of  $n = \frac{c}{v}$ ,  $v = \frac{1}{\sqrt{\epsilon\mu}}$ ,  $\epsilon = (1 + \tilde{\chi})\epsilon_0$ .

And for real, absorptive materials and complex representations of fields, we can use a complex  $n$ , with the imaginary part describing absorption:

$$n(\omega) = n_1(\omega) + i n_2(\omega)$$

It follows that we can write  $\tilde{\chi}$  as complex:

$$\tilde{\chi}(\omega) = \chi_1(\omega) + i \chi_2(\omega)$$

Now we have a complex-valued response function. And it's possible (though not easy) to show that a causal response function's Fourier transform must be analytic (in the complex math sense of the word). So  $\tilde{\chi}(\omega)$  is analytic.

As you may recall, analytic functions have strongly related real and imaginary parts. For example, given a function

$$f(x,y) = u(x,y) + i v(x,y)$$

for  $f$  to be analytic, we must have

$$\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = -\frac{dv}{dx}$$

But it gets better than that.

If you have some analytic  $\tilde{\chi}(\omega) = \chi_1(\omega) + i\chi_2(\omega)$  that falls off reasonably quickly as  $\omega \rightarrow \infty$ , it's actually possible to completely reconstruct either  $\chi_1$  or  $\chi_2$  from the other. The recipes for doing so are known as the Kramers-Kronig relations:

$$\chi_1(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega' \chi_2(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\chi_2(\omega) = -\frac{2\omega}{\pi} \int_0^{\infty} \frac{\chi_1(\omega')}{\omega'^2 - \omega^2} d\omega'$$

And since  $\chi_1$  basically represents dispersion while  $\chi_2$  represents absorption, we see that those two phenomena are fundamentally and inextricably linked.

Even better, it shows us that we can indirectly measure the real part of  $n(\omega)$  for some material by measuring its absorption spectrum.