EXAM 3 Solutions

Note Title 4/16/2008

1) Let $\hat{H}$ be a two-state Hamiltonian whose representation in some coord. system is

$$H = \begin{bmatrix} E_1 & \epsilon \\ \epsilon & E_2 \end{bmatrix}$$

a) compute the eigenvalues

$$\mathrm{Det}\begin{pmatrix} E_1-\lambda & \epsilon \\ \epsilon & E_2-\lambda \end{pmatrix} = (E_1-\lambda)(E_2-\lambda) - \epsilon^2 = 0$$

$$\Rightarrow (E_1-\lambda)(E_2-\lambda) = \epsilon^2$$

$$\Rightarrow \lambda^2 - \lambda(E_1+E_2) + E_1E_2 - \epsilon^2 = 0$$

$$\Rightarrow \lambda = \frac{(E_1+E_2)}{2} \pm \frac{1}{2}\sqrt{(E_1+E_2)^2 - 4(E_1E_2-\epsilon^2)}$$

$$\lambda_\pm = \frac{1}{2}(E_1+E_2) \pm \frac{1}{2}\sqrt{(E_1-E_2)^2 + 4\epsilon^2}$$

b) show that the presence of the off-diagonal term eliminates any possibility of degeneracy. Hint compute the difference in energy levels.

$$\lambda_+ - \lambda_- = \sqrt{(E_1-E_2)^2 + 4\epsilon^2}$$

The only way this can be zero is if a) $E_1 = E_2$ and $\epsilon = 0$

2. An electron is confined to a cube $[0,L]^3$ with ∞ potential at the boundary.

Fund. principle is schrödinger :

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) + V(x,y,z)\psi = E\psi$$

in our case we have

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \qquad \psi\Big|_{\text{Boundary}} = 0$$

$$\Rightarrow \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = -\frac{2mE}{\hbar^2}\psi$$

method of solution

apply seperation of variables

introduce appropriate sep. constants

apply boundary cond. at $x=0,L \quad y=0,L \quad z=0,L$

obtain stationary states

Sep. of variables $\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$\psi(x,y,z) = \sin\left(\frac{n_1 \pi x}{L}\right) \sin\left(\frac{n_2 \pi y}{L}\right) \sin\left(\frac{n_3 \pi z}{L}\right)$$

$$(n_1^2 + n_2^2 + n_3^2)\frac{\pi^2}{L^2} = \frac{2mE}{\hbar^2} \quad \text{(schröd. eqn)}$$

$$\boxed{\begin{gathered} E_{n_1,n_2,n_3} = \frac{\hbar^2 \pi^2}{2m L^2}(n_1^2 + n_2^2 + n_3^2) \\ \psi_{111} = N \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \sin\left(\frac{\pi z}{L}\right) \\ \text{normalization} \quad N = \left(\sqrt{\frac{2}{L}}\right)^3 \end{gathered}}$$

$$E_{n_1 n_2 n_3} = K(n_1^2 + n_2^2 + n_3^2) \qquad K = \frac{\hbar^2 \pi^2}{2mL^2}$$

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b)

N	n_1	n_2	n_3	E_{n_1,n_2,n_3}	deg.
1	1	1	1	3K	1
2	2	1	1	6K	3
3	1	2	1	6K	3
4	1	1	2	6K	3
5	2	2	1	9K	3
6	2	1	2	9K	3

7	1 2 2	9K	
8	3 1 1	11 K	3
9	1 3 1	11 K	3
10	1 1 3	11 K	3
11	2 2 2	12 K	1

3 Assume $Af = \alpha f \quad Bf = \mu f$

$$\Rightarrow \quad BAf = \alpha \underbrace{Bf}_{\mu f} \qquad ABf = \mu \underbrace{Af}_{\alpha f}$$

$$\Rightarrow \quad BAf = \alpha \mu f \qquad ABf = \mu \alpha f = \alpha \mu f$$

so $\quad BAf = ABf$

$$\Rightarrow \quad AB - BA = 0$$

$$\psi_1(x) = \sqrt{2}\,\sin(\pi x)$$

$$\phi_1(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^1 \sqrt{2}\,\sin(\pi x)\, e^{-ipx/\hbar}\, dx$$

$$\sin(\pi x) = \frac{e^{i\pi x} - e^{-i\pi x}}{2i}$$

$$\phi_1(p) = -\frac{i}{2} \frac{1}{\sqrt{\pi\hbar}} \int_0^1 e^{i(\pi - p/\hbar)x} - e^{-i(\pi + p/\hbar)x}\, dx$$

$$\int_0^1 e^{i\alpha x}\,dx = \frac{1}{i\alpha} e^{i\alpha x}\Big|_0^1 \qquad \alpha = \pi - p/\hbar$$

$$= \frac{1}{i\alpha}\left(e^{i\alpha} - 1\right)$$

$$\int_0^1 e^{-i\alpha' x}\,dx = \frac{-1}{i\alpha'}\left(e^{-i\alpha'} - 1\right) \qquad \alpha' = \pi + p/\hbar$$

$$\varphi_1(p) = -\frac{1}{2}\frac{1}{\sqrt{\pi\hbar}}\left[\underbrace{\frac{e^{i(\pi - p/\hbar)} - 1}{\pi - p/\hbar}} + \frac{e^{-i(\pi + p/\hbar)} - 1}{\pi + p/\hbar}\right]$$

$$\frac{-e^{-ip/\hbar} - 1}{\pi - p/\hbar} + \frac{-e^{-ip/\hbar} - 1}{\pi + p/\hbar}$$

$$\frac{1}{2}\frac{1}{\sqrt{\pi\hbar}}\left(e^{-ip/\hbar} + 1\right)\underbrace{\left[\frac{1}{\pi - p/\hbar} + \frac{1}{\pi + p/\hbar}\right]}$$

$$\frac{\pi + p/\hbar + (\pi - p/\hbar)}{\pi^2 - p^2/\hbar^2}$$

$$\frac{2\pi\hbar^2}{\hbar^2\pi^2 - p^2}$$

$$\varphi_1(p) = \frac{1}{\sqrt{\pi\hbar}}\,\frac{\pi\hbar^2}{\hbar^2\pi^2 - p^2}\left(1 + e^{-ip/\hbar}\right)$$

$$\boxed{\varphi_1(p) = \sqrt{\frac{\pi}{\hbar}}\,\frac{1}{\pi^2 - (p/\hbar)^2}\left(1 + e^{-ip/\hbar}\right)}$$

$$(1+e^{+ip/\hbar})(1+e^{-ip/\hbar}) = 2+2\cos(p/\hbar)$$

$$|\varphi_1|^2 = \frac{2\pi}{\hbar} \frac{(1+\cos(p/\hbar))}{(\pi^2-(p/\hbar)^2)^2}$$

$$\langle \varphi_1 | p^2 \varphi_1 \rangle = \frac{2\pi}{\hbar} \int_{-\infty}^{\infty} \frac{p^2(1+\cos(p/\hbar))}{(\pi^2 - p^2/\hbar^2)^2} dp$$

$p/\hbar \to k$

$p^2 = \hbar^2 k^2$

$dp = \hbar\, dk$

$$= 2\pi \hbar^2 \int_{-\infty}^{\infty} \frac{k^2(1+\cos(k))}{(\pi^2-k^2)^2} dk$$

5) $\psi_{100} = \frac{a_0^{-3/2}}{\sqrt{\pi}} e^{-r/a_0}$

Radial prob. density is

$$|\psi_{100}|^2 dV = |\psi_{100}|^2 4\pi r^2 dr$$

$$P(r)\,dr = \frac{4}{a_0^3} e^{-2r/a_0} r^2 dr$$

So $$P(r) = \frac{4}{a_0^3} e^{-2r/a_0} r^2$$

Set $\frac{dP}{dr} = 0$: $$\frac{4}{a_0^3}\left\{ \cancel{2} r \cancel{e^{-2r/a_0}} + r^2\left(-\frac{\cancel{2}}{a_0}\right)\cancel{e^{-2r/a_0}} \right\} = 0$$

$$\Rightarrow \quad r - \frac{r^2}{a_0} = 0$$

$$\Rightarrow \quad r = a_0$$ most probable r

You can verify for yourself that this is different than $\langle r \rangle$

$$\langle r \rangle = \frac{3}{2} a_0$$ expectation of r

6) a) if $[\hat{Q}, \hat{H}] = 0$ then

$$\frac{d\hat{Q}}{dt} = 0 \qquad \left(\text{assuming no explicit time depend. } \frac{\partial Q}{\partial t} = 0\right)$$

Thus an operator that commutes with the Hamiltonian represents a conserved quantity.

b) $13.6\,eV\left(1 - \frac{1}{2^2}\right) = 10.2\,eV$

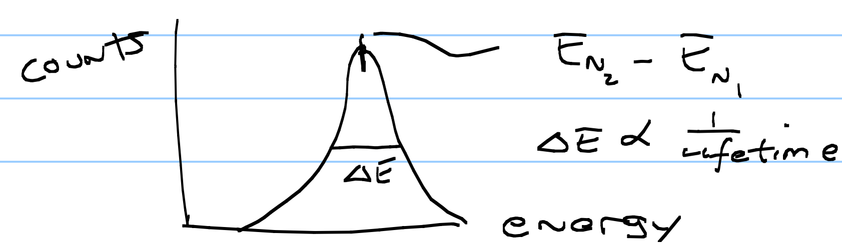
$= 121.6\ nm$ (UV)

$$\frac{3 \times 10^8\ m/s}{121.6 \times 10^{-9}\ m} = .025 \times 10^{17} = 2.5 \times 10^{15}\ Hz$$

7) a) in practice, excited states have finite lifetimes. Since $\Delta t\, \Delta E \geq \hbar/2$ this means

that there must be a corresponding spread of energies.

So



b) stationary states do not decay. There is no explanation of this in elementary QM. to go further we must understand the influence of vacuum fluctuations.