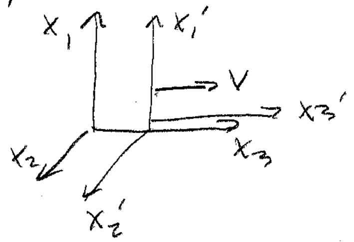
Relativistic electrodynamics

Einstein's postulates

1) all physical laws are the same in different inertial systems (systems moving at const. vel.)
2) c is a universal constant (in vacuum)

compare to Galilean transform (classical)



$$x_1' = x_1$$
$$x_2' = x_2$$
$$x_3' = x_3 - vt$$
$$t' = t$$

here, measurement of length <u>unchanged</u>:

$$(\Delta s)^2 = \sum_j (\Delta x_j)^2 = \sum_j (\Delta x_j')^2$$

is wave eqn. invariant to Galil. transform?

$$\frac{\partial}{\partial x_j} \to \frac{\partial}{\partial x_j'} \qquad \text{ex. } \frac{f(x_{j1}') - f(x_{j2}')}{x_{j1}' - x_{j2}'} = \frac{f(x_{j1}) - f(x_{j}}{x_{j1} - x_{j2}}$$

$$\frac{\partial}{\partial t} \psi(x_3'(t), t) = \frac{\partial x_3'}{\partial t}\frac{\partial \psi}{\partial x_3'} + \frac{\partial \psi}{\partial t}$$

$$= \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x_3'}\right)\psi$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x_3'}$$

$$\partial_t^2 \to \partial_{t'}^2 + v^2 \partial_{x_3'}^2 - 2v\frac{\partial^2}{\partial t' \partial x_3'}$$

$$\therefore \nabla^2\psi - \frac{1}{c^2}\partial_t^2\psi \to \nabla'^2\psi - \frac{1}{c^2}\partial_{t'}^2\psi - \frac{v^2}{c^2}\partial_{x_3'}^2\psi + \frac{2v}{c}\frac{\partial^2\psi}{\partial t'\partial x_3'}$$

ok at $v/c \ll 1$, not in general.

c = const. → consequences:
in vacuum, time for light to travel Δs is

$$t^2 = \frac{\sum (\Delta x_i)^2}{c^2}$$

set $\vec{r}=0$ at $t=0$, then if t is indep of frame,
$\sum x_i^2 - c^2 t^2 = 0$ in all frames.

define $x_4 = ict$
$\rightarrow \sum_1^4 x_\mu^2 = 0$

index notation: $x_\mu x_\mu = 0$
sum over repeated indices
greek 1 to 4
roman 1 to 3

so, for same wave, and
$\vec{r}' = 0$ at $t' = 0$

$$\sum x_\mu'^2 = 0$$

in general $\sum x_\mu^2 = \sum x_\mu'^2$ = constant.
other words, length of 4-vector is unchanged by transformation.

∴ coord transformation is a <u>rotation</u> in 4-D space.
"Minkowski space"
note: using ict is just a way to handle − sign.
other systems - co-variant, contra-variant ...

in general rotation is accompl. by a matrix:
$\vec{X}' = \overleftrightarrow{R}_\phi \cdot \vec{X}$

rotation around $\hat{z}$ → $R_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

index form:

$$X_i = R_{ij} X_j \qquad \text{sum on } j = 1,2,3$$

Lorentz transform:

$$X'_\mu = \lambda_{\mu\nu} X_\nu$$

this transform must preserve 4-D length:

$$X'_\mu X'_\mu = X_\nu X_\nu$$

$$= \lambda_{\mu\nu'} X_{\nu'} \lambda_{\mu\nu} X_\nu$$

if $\lambda_{\mu\nu'} \lambda_{\mu\nu} = \delta_{\nu\nu'}$ then length is preserved.

this means transf. matrix has orthonormal eigenvectors.
(general prop. of rotation matrix)

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} \qquad \begin{aligned} \vec{v}_1 \cdot \vec{v}_3 &= 0 \\ |v_1|^2 &= 1 \end{aligned}$$

Based on this form, we can derive the form of the Lorentz transform (see HW 14.2)

For v in z direction,

$$\overleftrightarrow{\lambda} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \gamma & i\beta\gamma \\ & & -i\beta\gamma & \gamma \end{pmatrix} \qquad \text{with } \begin{aligned} \beta &= v/c \\ \gamma &= (1-\beta^2)^{1/2} \end{aligned}$$

$$x_3' = \gamma x_3 + i\beta\gamma ct = \gamma(x_3 - ivt)$$

$$ict' = -i\beta\gamma x_3 + \gamma ict =$$

$$\rightarrow t' = \gamma(t - \beta/c\, x_3)$$

Lorentz transform as rotation (HM 14-3)

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\cos(i\alpha) = \frac{1}{2}(e^{-\alpha} + e^{+\alpha}) = \cosh\alpha$$

$$\sin(i\alpha) = \frac{1}{2i}(e^{-\alpha} - e^{+\alpha}) = i\sinh\alpha$$

$$\cos^2(i\alpha) + \sin^2(i\alpha) = 1 = \cosh^2\alpha - \sinh^2\alpha$$

let $\tanh\alpha = \beta = v/c$ $\qquad 1 - \tanh^2\alpha = 1 - \beta^2 = 1/\cosh^2\alpha$

then $\cosh\alpha = \dfrac{1}{\sqrt{1-\beta^2}} = \gamma$

$$\sinh\alpha = \sqrt{\cosh^2\alpha - 1}$$

$$= \sqrt{\frac{1}{1-\beta^2} - 1} = \frac{\beta}{\sqrt{1-\beta^2}} = \gamma\beta$$

$\therefore$ we can write λ as

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cosh\alpha & i\sinh\alpha \\ & & -i\sinh\alpha & \cosh\alpha \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos(i\alpha) & \sin(i\alpha) \\ & & -\sin(i\alpha) & \cos(i\alpha) \end{pmatrix}$$

Space-time diagrams.

event A at $(x_{1A}, x_{2A}, x_{3A}, x_{4A})$

event B at $(x_{1B} - - - -)$

displacement 4-vector: $\Delta x_\mu = x_{\mu A} - x_{\mu B}$

interval = $I = \Sigma(\Delta x_\mu)^2 = d^2 - c^2t^2$

if $I = 0$ → "light like"

e.g. A = flash at $x_\mu = 0$

B = detect at $x_{\mu B}$

$I > 0$ → "spacelike"

$|d| > ct$ A cannot cause B

$I < 0$ → timelike

$|d| < ct$ possible causality