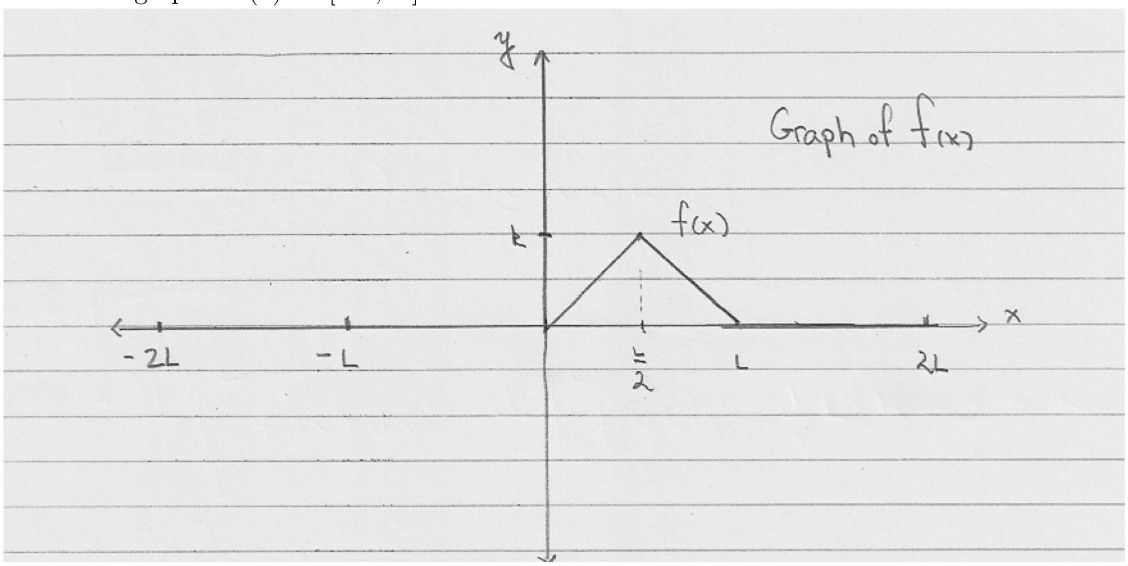
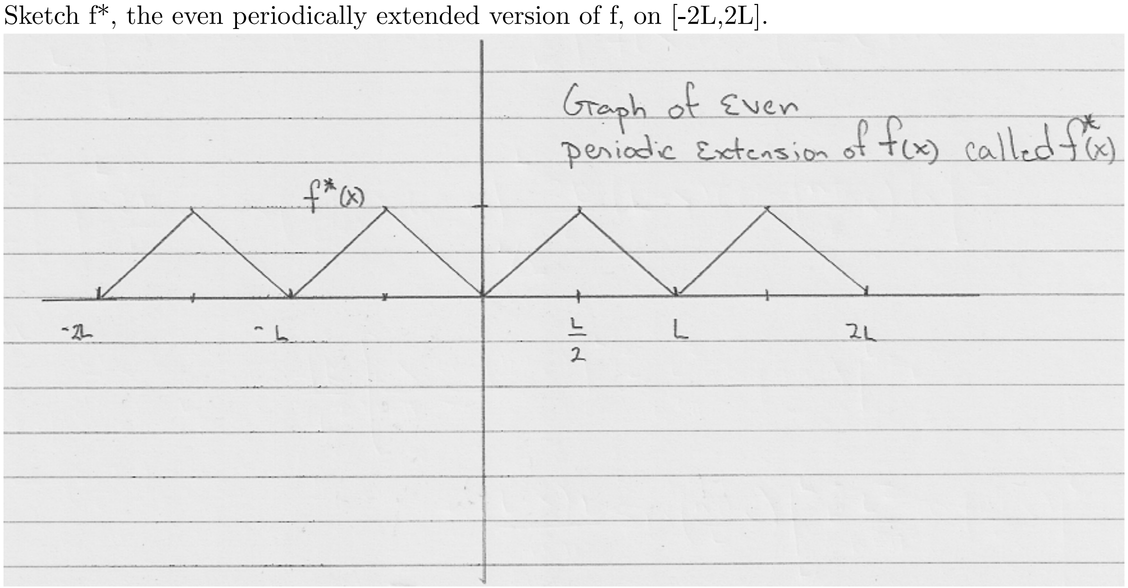
1. Given

$$f(x) = \begin{cases} \dfrac{2k}{L}x & 0 \le x \le \frac{L}{2}, \\ \\ \dfrac{2k}{L}(L-x) & \frac{L}{2} < x \le L. \end{cases} \tag{1}$$

(a) Sketch the graph of f(x) on [-2L,2L].



(b) Sketch f*, the even periodically extended version of f, on [-2L,2L].



(c) Calculate the Fourier series of f^*. That is calculate the Fourier cosine half-range expansion of f.

We begin with the coefficient a_0 and not hing that this is nothing more than the area under the curve $f(x)$ we find,

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f^*(x)dx && (2)\\
&= \frac{1}{L}\int_{0}^{L} f(x)dx && (3)\\
&= \frac{1}{L}\cdot\frac{1}{2}Lk && (4)\\
&= \frac{k}{2} && (5)
\end{aligned}$$

Next we have a_n. Some symmetry, integration by parts and algebra gives,

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f^*(x)\cos\left(\frac{n\pi}{L}x\right)dx && (6)\\
&= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx && (7)\\
&= \frac{2}{L}\left[\frac{2k}{L}\int_{0}^{\frac{L}{2}} x\cos\left(\frac{n\pi}{L}x\right)dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\left(\frac{n\pi}{L}x\right)dx\right] && (8)\\
&= \frac{4k}{L^2}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Big|_0^{\frac{L}{2}} + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\Big|_0^{\frac{L}{2}} + \frac{L}{n\pi}(L-x)\sin\left(\frac{n\pi}{L}x\right)\Big|_{\frac{L}{2}}^{L} - \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\Big|_{\frac{L}{2}}^{L}\right] && (9)\\
&= \frac{4k}{L^2}\left[\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}x\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right) - \frac{L^2}{n^2\pi^2} - \frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}x\right) - \frac{L^2}{n^2\pi^2}(-1)^n + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right)\right] && (10)\\
&= \frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}x\right) - (-1)^n - 1\right] && (11)
\end{aligned}$$

Further simplifications can be made. If we note the following pattern,

$$\begin{aligned}
n = 1 &\Rightarrow a_1 = 0, && (12)\\
n = 2 &\Rightarrow a_2 = -\frac{16k}{2^2 n^2}, && (13)\\
n = 3 &\Rightarrow a_3 = 0, && (14)\\
n = 4 &\Rightarrow a_4 = 0, && (15)\\
n = 5 &\Rightarrow a_5 = 0, && (16)\\
n = 6 &\Rightarrow a_6 = -\frac{-16k}{6^2 n^2}, && (17)
\end{aligned}$$

we can write the Fourier cosine series as,

$$\begin{aligned}
f(x) &= \frac{k}{2} + \sum_{n=1}^{\infty}\frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}x\right) - (-1)^n - 1\right]\cos\left(\frac{n\pi}{L}x\right) && (18)\\
&= \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\left(\frac{2\pi}{L}x\right) + \frac{1}{6^2}\cos\left(\frac{6\pi}{L}x\right) + \cdots\right) && (19)
\end{aligned}$$

2. (a) Show that $\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = 2\pi\delta_{mn}$ where $m, n \in R$.

For $n \neq m$
$$\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \int_{-\pi}^{\pi} e^{(n-m)ix}dx = \frac{e^{(n-m)ix}}{i(n-m)}\Big|_{-\pi}^{\pi} =$$
$$= \frac{(-1)^{(n-m)}}{i(n-m)} - \frac{(-1)^{(n-m)}}{i(n-m)} = 0$$

For $n = m$
$$\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \int_{-\pi}^{\pi} e^{(n-m)ix}dx = \int_{-\pi}^{\pi} 1dx = x|_{-\pi}^{\pi} = 2\pi$$
$$= \int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases} = 2\pi\delta_{nm}$$

(b) Find the Fourier Coefficients of $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
&\Rightarrow f(x)e^{-imx} = \sum_{n=-\infty}^{\infty} c_n e^{inx} e^{-imx} \\
&\Rightarrow \int_{-\infty}^{\infty} f(x)e^{-imx} dx = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} c_n e^{(n-m)x} dx \\
&\text{As we found in (a), the integral on the right is 0 for all values of n except n=m} \\
&\Rightarrow \int_{-\pi}^{\pi} f(x)e^{-imx} dx = c_m 2\pi \\
&\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-imx} dx = c_m \\
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx
\end{aligned}
$$

Because m=n we can replace our m's with n's to get the formula for c_n

3. Let $f(x) = x^2 \; -\pi < x < \pi$ be 2π periodic.

(a) Calculate the complex Fourier Series representation of f(x).

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx \\
&= \frac{1}{2\pi} \left[\frac{-x^2}{in} e^{-inx} + \frac{2x}{n^2} e^{-inx} + \frac{2}{in^3} e^{-inx} \right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left[\left(\frac{-x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3} \right) e^{-inx} \right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left[\left(\frac{-\pi^2}{in} + \frac{2\pi}{n^2} + \frac{2}{in^3} + \frac{\pi^2}{in} + \frac{2\pi}{n^2} - \frac{2}{in^3} \right) (-1)^n \right] \\
&= \frac{1}{2\pi} \left[\frac{4\pi}{n^2} (-1)^n \right] = \frac{2}{n^2} (-1)^n \qquad n \neq 0 \\
&\text{For } n = 0 \\
c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{i(0)x} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \left[\frac{1}{3} x^2 \right]_{-\pi}^{\pi} = \\
&= \frac{\pi^2}{3} \\
f(x) &= \frac{\pi^2}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{2}{n^2} (-1)^n e^{inx}
\end{aligned}
$$

(b) Using (a), recover the real Fourier series representation of f(x).

$$
\begin{aligned}
f(x) &= \frac{\pi^2}{3} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{2}{n^2}(-1)^n e^{inx} \\
&= \frac{\pi^2}{3} + \sum_{n=-\infty}^{-1} \frac{2}{n^2}(-1)^n e^{inx} + \sum_{1}^{\infty} \frac{2}{n^2}(-1)^n e^{inx} \\
&\text{Substituting n=-n into the first series we get:} \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n e^{-inx} + \sum_{1}^{\infty} \frac{2}{n^2}(-1)^n e^{inx} \\
&= \frac{\pi^2}{3} + \sum_{1}^{\infty} \frac{2}{n^2}(-1)^n \left(e^{-inx} + e^{inx}\right) \\
&\text{Using Euler's Formula:} \\
&= \frac{\pi^2}{3} + \sum_{1}^{\infty} \frac{2}{n^2}(-1)^n \left(cos(nx) - isin(nx) + cos(nx) + sin(nx)\right) \\
&\text{The Real Fourier Series Representation:} \\
f(x) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n cos(nx)
\end{aligned}
$$

4. (a) How is the Fourier transform related to Fourier Series? You should discuss both the periodicity and number of Fourier modes used in the construction of each.

Fourier transforms and Fourier series are the same in that they take a function and represent it as the sum of oscillatory functions multiplied by amplitudes of oscillations. However, they differ by the number of Fourier modes (terms in the summations) used. In the case of Fourier series there are countably infinite number of oscillatory functions which depend on a countably infinite frequency spectrum and these modes are used to construct periodic functions. In the case of Fourier transforms the spectrum must be continuous and thus there are uncountably infinite modes depending on a continuum of frequencies.

(b) What does cross-correlation measure? What would auto-correlation measure?

Correlation is a measure of similarity between two functions. This measure is given in terms of a convolution integral. If cross-correlation is the measure of similarity between two functions then auto-correlation is a measure between the function and itself.

(c) What is the uncertainty principle as it relates to Fourier transforms? How much power would be required to send a signal like $\delta(t)$?

The uncertainty principle for Fourier transforms says that if a function is localized in one domain then it is de-localized in the transformed domain. One can interpret this physically as saying if you know very well the spread of the function in one space then you know very little about the spread in another space. These two spaces are typically position-momentum or time-frequency. Using this idea one can show that the Fourier transform of a delta function is a constant function and the power of the signal is the area under the square of this constant function, which is infinite. Therefore you would need an infinite amount of power to send a single impulse of signal.

5.

$$f(x) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw \tag{20}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx \tag{21}$$

$$f_c(x) = \sqrt{\tfrac{2}{\pi}} \int_0^{\infty} \hat{f}(w) cos(wx) dw \qquad \hat{f}_c(w) = \sqrt{\tfrac{2}{\pi}} \int_0^{\infty} f(x) cos(wx) dx \tag{22}$$

$$f_s(x) = \sqrt{\tfrac{2}{\pi}} \int_0^{\infty} \hat{f}(w) sin(wx) dw \qquad \hat{f}_s(w) = \sqrt{\tfrac{2}{\pi}} \int_0^{\infty} f(x) sin(wx) dx \tag{23}$$

(a) Show that $f_c(x)$ and $\hat{f}_c(w)$ are even functions and $f_s(x)$ and $\hat{f}_s(w)$ are odd functions.

In $f_c(x)$, cos(wx) is the only function of x, so that $f_c(-x)$=$f_c(x)$, the definition of an even funtion. Similarly, $\hat{f}_c(-w)$=$\hat{f}_c(w)$.

In $f_s(x)$, sin(wx) is the only function of x, so that $f_s(-x)$=$-f_s(x)$, the definition of an odd function. Similarly, $\hat{f}_s(-w)$=$-\hat{f}_s(w)$.

(b) Show that, if we assume f(x) is even, (1)-(2) defines the transform pair (3) and if we assume f(x) is odd, (1)-(2) defines the transform pair (4).

Using Euler's Formula, (2) becomes

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(cos(wx) - isin(wx))dx$$
$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)cos(wx)dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)sin(wx)dx$$

If we assume f(x) is even, the imaginary part is an odd function integrated over symetric bounds, which becomes zero and $\hat{f}(w) = \hat{f}_c(w)$.

Because $\hat{f}(w) = \hat{f}_c(w)$, which is even, the same argument can be used to show that $f(x) = f_c(x)$.

If we assume f(x) is odd

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$
$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)cos(wx)dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)sin(wx)dx$$

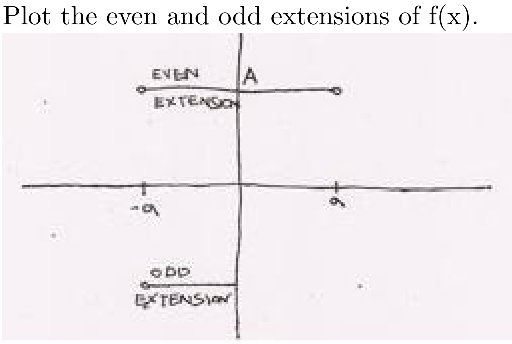
Becuase f(x) is odd, the real part becomes an odd function integrated over symetric bounds which becomes zero and $\hat{f}(w) = \hat{f}_s(w)$.

Becuase $\hat{f}(w) = \hat{f}_s(w)$, which is odd, the same argument can be used to show that $f(x) = f_s(x)$.

(c)

$$f(x) = \begin{cases} A & 0 < x < a \\ 0 & \text{otherwise} \end{cases} \qquad A, a \in R^+ \tag{24}$$

Plot the even and odd extensions of f(x).



(d) Find the Fourier Cosine and Sine transforms of f(x).

$$\begin{aligned}
\hat{f}_c(w) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cos(wx)dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^a A\cdot cos(wx)dx = \sqrt{\frac{2}{\pi}}\left[\frac{A}{w}sin(wx)\right]_0^a \\
\hat{f}_c(w) &= \sqrt{\frac{2}{\pi}}\left(\frac{A\cdot sin(aw)}{w}\right) \\
\hat{f}_s(w) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sin(wx)dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^a A\cdot sin(wx)dx = \sqrt{\frac{2}{\pi}}\left[\frac{-A}{w}cos(wx)\right]_0^a \\
\hat{f}_s(w) &= -\sqrt{\frac{2}{\pi}}\left(\frac{A(cos(aw)-1)}{w}\right)
\end{aligned}$$

(e) Using the Fourier Cosine transform, show that $\int_{-\infty}^{\infty}\frac{sin(w\pi)}{w\pi}dw = 1$.

From (d) we have that if $f(x) = \begin{cases} A & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\left(\frac{A\cdot sin(wa)}{w}\right)$$

If we take the inverse transform

$$F^{-1}\left\{\hat{f}_c(w)\right\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty\left[\sqrt{\frac{2}{\pi}}\left(\frac{A\cdot sin(aw)}{w}\right)\right]cos(wx)dw$$

Evaluating f(x) at 0

$$f(0) = \frac{2}{\pi}\int_0^\infty\frac{A\cdot sin(wa)}{w}dw = A$$

Because $\frac{A\cdot sin(wa)}{w}$ is an even function,

$$\frac{2}{\pi}\int_0^\infty\frac{A\cdot sin(wa)}{w}dw = \frac{1}{\pi}\int_{-\infty}^\infty\frac{A\cdot sin(wa)}{w}dw = A$$

If we choose A=1 and $a = \pi$, $\int_{-\infty}^{\infty}\frac{sin(w\pi)}{w\pi}dw = 1$