MACS 332 - August 9, 2006
Exam III - 60 minutes

NAME: Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. Please enclose your final answers in boxes.

1. (10 Points) Briefly describe the following mathematical concepts.

a. The norm of a vector in $\mathbb{R}^n$.

square root of the

The norm of a vector is the inner product of that vector with itself.

The norm of a vector is the length of that vector.

b. The Gram-Schmidt Process.

Basis

The gram-schmit process take a set of linearly independent vectors and forms an orthogonal set of vectors. It works by taking orthogonal projections of vectors into subspaces spanned by previous vectors.

Basis

c. The general least-squares problem.

The general least-squares problem is the problem of finding an $\vec{x}$ s.t. $\|\vec{b} - A\vec{x}\|$ is minimized for the inconsistent system $A\vec{x} = \vec{b}$.

d. The matrix $\mathbf{U}_{n \times n}$ where $\mathbf{U}$ is such that $\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = \mathbf{I}$.

U is an orthogonal matrix. That is U has orthonormal columns and its inverse is equal to its transpose.

2. (10 Points) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

a. Determine the least-squares solution of **Ax=b**.

Note since the columns of A are linearly independent we can use $\hat{x} = R^{-1}Q^T\vec{b}$ where Q^T is ...

(1) $Q^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow Q^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = R$

Thus $\hat{x} = R^{-1}Q^T\vec{b} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

(2) or

$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad [A^TA \mid \vec{b}] = \left[\begin{array}{cc|c} 2 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 3/2 \\ x_2 = 1 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b. Determine the least-squares error associated with the least-squares solution in (**a**).

$\|\vec{b} - \hat{b}\| = \|\vec{b} - A\hat{x}\| = \left\|\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}\right\| = \sqrt{1/2} = 1/\sqrt{2}$

3. (10 Points) Let

$$\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Calculate the distance from $\mathbf{y}$ to the plane W=span{$\mathbf{u}_1$,$\mathbf{u}_2$}.

$$\hat{y} = \frac{\vec{y}^T\vec{u}_1}{\vec{u}_1^T\vec{u}_1}\vec{u}_1 + \frac{\vec{y}^T\vec{u}_2}{\vec{u}_2^T\vec{u}_2}\vec{u}_2 = \frac{-1+4}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1+4}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$\|z\| = \|\vec{y} - \hat{y}\| = \left\| \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\| = 3$$

4. (10 Points) **Note** - Problem 4b is on page 4.

a. Let Q(**x**)=$\mathbf{x}^T\mathbf{A}\mathbf{x}$, where **A**= $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Determine the change of variable matrix **P** and diagonal matrix **D** such that Q(**x**)=Q(**y**)=$\mathbf{y}^T\mathbf{D}\mathbf{y}$, where $\mathbf{y}=\mathbf{P}^T\mathbf{x}$.

$$\det(A - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$\lambda = 1$:

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow x_1 = x_2 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

x_2 is free

$\lambda = -1$.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow x_1 = -x_2 \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

x_2 free

normalizing we get

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$Q(\vec{y}) = y_1^2 - y_2^2$$

b. Let Q(**x**)= $3x_2^2 - 4x_1x_2 + 6x_1^2$. Determine the symmetric two-by-two matrix **A**, such that Q(**x**)=$\mathbf{x}^T\mathbf{A}\mathbf{x}$.

$Q(\vec{x}) = [x_1\ x_2]\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$

5. (10 Points) Let **A**= $\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$. Determine the matrices **U**,Σ,**V**, associated with **A**=**U**Σ$\mathbf{V}^T$.

$A^TA = \begin{bmatrix} 49 & 0 \\ 0 & 0 \end{bmatrix}$ $\det(A^TA - \lambda I) = 0 \Rightarrow$ $\lambda = 49$, $\lambda = 0$

$\underline{\lambda = 49:}$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -49 & 0 \end{array}\right] \Rightarrow$ x_1 is free, $x_2 = 0$ $\Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\Rightarrow$ $\Sigma_{2\times 2} = \begin{bmatrix} \sqrt{49} & 0 \\ 0 & 0 \end{bmatrix}$

$\underline{\lambda = 0:}$

$\left[\begin{array}{cc|c} 49 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ $x_1 = 0$, $x_2 =$ free $\Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Finding U.

$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{49}}\begin{bmatrix} 7 \\ 0 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2$; undefined

Thus let $\vec{x}$ be such that, $\quad \Rightarrow U = \begin{bmatrix} 7/\sqrt{49} & 0 \\ 0 & 1 \end{bmatrix}$.

$\vec{u}_1^T\vec{x} = 0 = \frac{7}{\sqrt{49}}x_1 + 0x_2 = 0$

$\Rightarrow$ $x_1 = 0$, x_2 is free. Choose $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$