

Quote of Lecture 10	
Juliet: What's in a name? That which we call a rose by any other name would smell as sweet.	
	Shakespeare : Romeo and Juliet (1591)

1. REVIEW

So, at this point we have the following,

$$(1) \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right),$$

$$(2) \quad a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx,$$

$$(3) \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx,$$

$$(4) \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx,$$

which defines the Fourier series and it's associated coefficients for a $2L$ -periodic function, where L is a scaling parameter introduced to control the length of the period. We also have the following important results:

- Any function for, which the integrals (2)-(4) are defined has a Fourier series representation. Notice that this does not require the function to be periodic, but the Fourier series will induce this function to be periodic with principle domain $(-L,L)$.
- The Fourier series may actually differ from the function $f(x)$ at a countably infinite amount of points. We can know where this might occur by knowing the jump-discontinuities of f and we have that the Fourier series will average the right and left hand limits at these points.
- The Fourier series represents the function f in terms of it's oscillatory features for which the data f supplies the amplitudes for each oscillatory mode. ¹
- The Fourier series represents the function f in terms of it's even components and odd components. ²

2. LECTURE OVERVIEW

Now we are going to make use of the well celebrated Euler's formula,

$$(5) \quad e^{i\theta} = \cos(\theta) + i \sin(\theta), \quad i = \sqrt{-1},$$

so that we can rewrite (1)-(4) in its complex form,

$$(6) \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi}{L}x},$$

$$(7) \quad c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{i\frac{n\pi}{L}x} dx,$$

¹Each term in the series is called a Fourier mode and the lowest order term is often called the Fundamental mode.

²If the function f has symmetry then the equations (1)-(4) simplify according to the initial properties of symmetric functions.

which is tidy but lacks some of the clarity of the real-form. ³ From this form one can always derive the real Fourier series form and moreover if the function f is symmetric then this immediately simplifies to a Fourier cosine or Fourier sine series. The following outlines some pros and cons:

Pro: We need only remember 2 formula instead of 4.

Pro: Integrations involving exponential functions greatly simplify.

Con: The case for when n is often a special case (notice that $c_0 = a_0$) where the coefficient becomes singular due to anti-differentiation of the exponential function.

Con: From the complex form the graph of the periodic function is not as accessible.

Lastly, to calculate the energy in a ‘signal’ we note that the energy of a sinusoid is proportional the square of it’s amplitude ⁴ then we can conclude that the energy of a signal can be found by it’s Fourier coefficients as

$$(9) \quad E \propto a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2,$$

however in (6)-(7) the Fourier coefficients may be complex and the connection to energy is not as clear. In this case we have the following:

$$(10) \quad E \propto \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where $|c_n|^2 = c_n \bar{c}_n$. ⁵

3. LECTURE GOALS

Our goals with this material will be:

- Understand the connections both similarities and differences between complex and real Fourier series representations of functions.

4. LECTURE OBJECTIVES

The objectives of these lessons will be:

- Derive the complex Fourier series using the real Fourier series and associated coefficients.
- Learn to convert the complex Fourier series into a real Fourier series through algebraic simplifications.

³The coefficients, which we derive from a_n and b_n in class, can also be derived from the following orthogonality relation:

$$(8) \quad \left\langle e^{-i \frac{n\pi}{L}}, e^{-i \frac{m\pi}{L}} \right\rangle = 2L \delta_{nm}$$

⁴<http://www.glenbrook.k12.il.us/gbssci/phys/Class/waves/u1012c.html>

⁵Here the ‘bar’ denotes complex conjugation. If $z = \alpha + \beta i$ then $\bar{z} = \alpha - \beta i$ and one can easily conclude that $z\bar{z} \in \mathbb{R}$ as we would expect for a quantity like energy.