Day 24: The magnetic dipole

A long, long time ago we learned how to do multipole expansions for V:

$$V(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}}{r^2} + \frac{\hat{r}\cdot Q_2\cdot\hat{r}}{r^3}\right]$$

Where Q is the monopole moment, $\vec{p}$ the dipole moment, and Q_2 the quadrupole moment vector.

We can do a similar expansion of $\vec{A}$. We can reasonably expect one major difference, though. There are no magnetic monopoles, so the leading term in the expansion should be the dipole term.

We have $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{J(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$

And as with the V expansion, we start from

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\hat{r}\cdot\vec{x}'}{r^2} + \mathcal{O}\left(\frac{r'^2}{r^3}\right) \quad \text{with } r = |\vec{x}| \text{ and } r' = |\vec{x}'| \ll r$$

We're going to keep this expansion short & clean, so we keep only the first two terms. That gives us

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int \vec{J}(\vec{x}')\,d^3x' + \frac{1}{r^2}\int \vec{J}(\vec{x}')(\hat{r}\cdot\vec{x}')\,d^3x'\right]$$

In magnetostatics, $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \Rightarrow \vec{\nabla}\cdot\vec{J} = 0$ ie, current doesn't appear out of nowhere or disappear; $\vec{J}$ has no divergence. Only complete circuits of current exist. Thus, it should be conceptually palatable that

$$\int \vec{J}(\vec{x}')\,d^3x' = 0 \quad \text{(try writing in terms of divergence)}$$

We'll prove it, too: we note that

$$\vec{\nabla}\cdot(x_i\vec{J}) = \vec{J}\cdot(\nabla x_i) + (\vec{\nabla}\cdot\vec{J})\,x_i \qquad \text{And } \vec{\nabla}\cdot\vec{J} = 0, \quad \nabla x_i = \hat{e}_i$$

So $J_i = \vec{\nabla}\cdot(x_i\vec{J})$ and $\int J_i(\vec{x}')\,d^3x' = \int \vec{\nabla}\cdot(x_i\vec{J})\,d^3x' = \oint (x_i\vec{J})\cdot d\vec{A}$

And since the integral is over all space, as long as $\vec{J}$ falls off faster than $1/r$, the integrand is zero at the "area" bounding infinity.

$$\Rightarrow \int \vec{J}(\vec{x}')\,d^3x' = 0$$

So the monopole term in the multipole expansion is zero, leaving us with what must be the dipole term:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi r^2}\int J(\vec{x}')(\hat{r}\cdot\vec{x}')\,d^3x'$$

As with the voltage expansion, we need to decide how to write the dipole moment.

There, we wrote $\frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\,\hat{r}\cdot\vec{p}$

Here, we'll write $\frac{\mu_0}{4\pi}\cdot\frac{1}{r^2}\cdot\vec{m}\times\hat{r}$

Very analogous. We just have to figure out what $\vec{m}$ is from that $\vec{J}$ integral.

The derivation is lengthy & is in the book. The result is:

$$\vec{m} = \frac{1}{2}\int \vec{x}'\times J(\vec{x}')\,d^3x'$$

At which point

$$A(\vec{x}) = \frac{\mu_0}{4\pi r^2}\,\vec{m}\times\hat{r}$$

The B-field is

$$\boxed{\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\frac{[3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}]}{r^3}}$$

Optional derivation: $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\nabla\times\left(\vec{m}\times\frac{\hat{r}}{r^2}\right)$ Using an identity:

$$= \frac{\mu_0}{4\pi}\Bigg[\underbrace{\cancel{m\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right)}}_{1} - \underbrace{\cancel{\frac{\hat{r}}{r^2}\left(\vec{\nabla}\cdot\vec{m}\right)}}_{2} + \underbrace{\cancel{\left(\frac{\hat{r}}{r^2}\cdot\vec{\nabla}\right)\vec{m}}}_{3} - \underbrace{\left(\vec{m}\cdot\vec{\nabla}\right)\frac{\hat{r}}{r^2}}_{4}\Bigg]$$

1 is zero everywhere but at $r=0$ (the location of the dipole), since $\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\delta(r)$. And since the field is going to be divergent at $r=0$ anyway, this term contributes nothing useful.

2 and 3 involve derivatives of $\vec{m}$, which is a constant vector.

Only 4 matters. What does $(\vec{m}\cdot\vec{\nabla})$ mean? It's only easy to express in Cartesian:

$$(\vec{m}\cdot\vec{\nabla})\vec{f} = \left(m_x\frac{\partial}{\partial x}\right)f_x\,\hat{i} + \left(m_y\frac{\partial}{\partial y}\right)f_y\,\hat{j} + \left(m_z\frac{\partial}{\partial z}\right)f_z\,\hat{k}$$

$$\text{And so}\quad \left[(\vec{m}\cdot\vec{\nabla})\overbrace{\frac{\hat{r}}{r^2}}^{\vec{r}/r^3}\right]_x = m_x \frac{\partial}{\partial x}\overbrace{\frac{x}{r^3}}^{\text{part of } \vec{r}/r^3,\text{ same as } \hat{r}/r^2} = m_x \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= m_x\left[\frac{1}{r^3} + x\cdot(-3/2)\,r^{-5}\cdot 2x\right]$$

$$= \frac{m_x}{r^3} - \frac{3x^2 m_x}{r^5}$$

Using $m_x x = (\vec{m}\cdot\vec{r})_x$ and $x = r_x$

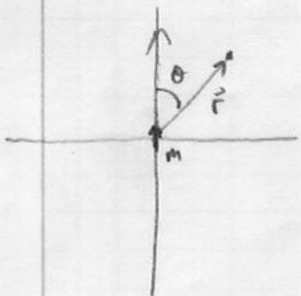
$$= \frac{m_x}{r^3} - \frac{3(\vec{m}\cdot\vec{r})_x r_x}{r^5}$$

Thus $$(\vec{m}\cdot\vec{\nabla})\frac{\hat{r}}{r^2} = \frac{\vec{m}}{r^3} - \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} = \frac{\vec{m}}{r^3} - \frac{3\hat{r}(\vec{m}\cdot\hat{r})}{r^3}$$

With the overall minus and the constants: $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}}{r^3}\right]$

We can clean up $\vec{A}$ & $\vec{B}$ by letting $\vec{m}$ point in the $\hat{k}$ direction (or line up the z-axis with $\vec{m}$)

Then, in spherical: $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi r^2}\, m\hat{k}\times\hat{r}$ And $\hat{k}\times\hat{r} = |\hat{k}||\hat{r}|\sin\theta\,\hat{\phi} = \sin\theta\,\hat{\phi}$



$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0 \sin\theta\,\hat{\phi}}{4\pi r^2}}$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[3\hat{r}(m\overbrace{\hat{k}\cdot\hat{r}}^{\cos\theta}) - m\hat{k}\right] \qquad \text{And } \hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

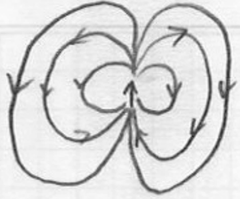
$$= \frac{\mu_0}{4\pi r^3}\left[3m\cos\theta\,\hat{r} - m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}\right]$$

$$\boxed{\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi r^3}\left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right]}$$

What does this look like? I'd have no idea without the help of a computer, except for the fact that the form matches that of the electric dipole:

$$\vec{E}(\vec{x}) = \frac{p_0}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \qquad (3.101,\ p.\ 78)$$

Which looks like :



Or rather like the field of a bar magnet, everyone's first dipole.

The parallels between electric and magnetic dipoles are legion:

	Electric	Magnetic
Dipole moment, general	$\vec{p} = \int \vec{x}' \rho(\vec{x}') d^3x'$	$\vec{m} = \frac{1}{2} \int \vec{x}' \times \vec{J}(\vec{x}') d^3x'$
Dipole moment, basic	$\vec{p} = q\vec{d}$ (two charges $\pm q$ separation $\vec{d}$)	$\vec{m} = I\vec{A}$ (current loop of area $\vec{A}$)
Potential	$V(\vec{x}) = \dfrac{\vec{p} \cdot \hat{r}}{4\pi\varepsilon_0 r^2}$	$\vec{A}(\vec{x}) = \dfrac{\mu_0}{4\pi} \dfrac{\vec{m} \times \hat{r}}{r^2}$
Field of	$\vec{E}(\vec{x}) = \dfrac{3\hat{r}(\vec{p} \cdot \hat{r}) - \vec{p}}{4\pi\varepsilon_0 r^3}$	$\vec{B}(\vec{x}) = \dfrac{\mu_0}{4\pi r^3}\left[3\hat{r}(\vec{m} \cdot \hat{r}) - \vec{m}\right]$
Torque on	$\vec{N} = \vec{p} \times \vec{E}$	$\vec{N} = \vec{m} \times \vec{B}$
Energy of	$U = -\vec{p} \cdot \vec{E}$	$U = -\vec{m} \cdot \vec{B}$

Pretty much the same stuff across the board. What's really different about the E + B cases is that for B, the dipole appears to be the most basic building block, whereas for E the monopole is.