Problem 5.18

$\boxed{\text{It doesn't matter.}}$ According to Theorem 2, in Sect. 1.6.2, $\int \mathbf{J}\cdot d\mathbf{a}$ is independent of surface, for any given boundary line, provided that $\mathbf{J}$ is divergenceless, which it *is*, for steady currents (Eq. 5.31).

Problem 5.25

(a) $\mathbf{A}$ points in the same direction as I, and is a function only of s (the distance from the wire). In cylindrical coordinates, then, $\mathbf{A} = A(s)\,\hat{\mathbf{z}}$, so $\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A}{\partial s}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$ (the field of an infinite wire). Therefore $\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi s}$, and $\boxed{\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{2\pi}\ln(s/a)\,\hat{\mathbf{z}}}$ (the constant a is arbitrary; you could use 1, but then the units look fishy). $\nabla\cdot\mathbf{A} = \frac{\partial A_z}{\partial z} = 0.$ ✓ $\nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} = \mathbf{B}.$ ✓

(b) Here Ampére's law gives $\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi s = \mu_0 I_{\text{enc}} = \mu_0 J\,\pi s^2 = \mu_0\frac{I}{\pi R^2}\pi s^2 = \frac{\mu_0 I s^2}{R^2}$.
$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{Is}{R^2}\hat{\boldsymbol{\phi}}$. $\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi}\frac{s}{R^2} \Rightarrow \mathbf{A} = -\frac{\mu_0 I}{4\pi R^2}(s^2-b^2)\,\hat{\mathbf{z}}$. Here b is again arbitrary, except that since $\mathbf{A}$ must be continuous at R, $-\frac{\mu_0 I}{2\pi}\ln(R/a) = -\frac{\mu_0 I}{4\pi R^2}(R^2-b^2)$, which means that we must pick a and b such that $2\ln(R/b) = 1-(b/R)^2$. I'll use $a=b=R$. Then

$$\boxed{\mathbf{A} = \begin{cases} -\frac{\mu_0 I}{4\pi R^2}(s^2-R^2)\,\hat{\mathbf{z}}, & \text{for } s\le R; \\ -\frac{\mu_0 I}{2\pi}\ln(s/R)\,\hat{\mathbf{z}}, & \text{for } s\ge R. \end{cases}}$$

Problem 5.27

(a) $\nabla\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau'$. $\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J}) + \mathbf{J}\cdot\nabla\left(\frac{1}{\mathscr{r}}\right)$. But the first term is zero, because $\mathbf{J}(\mathbf{r}')$ is a function of the *source* coordinates, not the *field* coordinates. And since $\boldsymbol{\mathscr{r}} = \mathbf{r}-\mathbf{r}'$, $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\nabla'\left(\frac{1}{\mathscr{r}}\right)$. So $\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = -\mathbf{J}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$. But $\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J}) + \mathbf{J}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)$, and $\nabla'\cdot\mathbf{J} = 0$ in magnetostatics (Eq. 5.31). So $\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = -\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)$, and hence, by the divergence theorem, $\nabla\cdot\mathbf{A} = -\frac{\mu_0}{4\pi}\int\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau' = -\frac{\mu_0}{4\pi}\oint\frac{\mathbf{J}}{\mathscr{r}}\cdot d\mathbf{a}'$, where the integral is now over the *surface* surrounding all the currents. But $\mathbf{J}=0$ on this surface, so $\nabla\cdot\mathbf{A} = 0.$ ✓

(b) $\nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\int\nabla\times\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau' = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}(\nabla\times\mathbf{J}) - \mathbf{J}\times\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$. But $\nabla\times\mathbf{J} = 0$ (since $\mathbf{J}$ is not a function of $\mathbf{r}$), and $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$ (Eq. 1.101), so $\nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau' = \mathbf{B}.$ ✓

(c) $\nabla^2\mathbf{A} = \frac{\mu_0}{4\pi}\int\nabla^2\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau'$. But $\nabla^2\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \mathbf{J}\nabla^2\left(\frac{1}{\mathscr{r}}\right)$ (once again, $\mathbf{J}$ is a *constant*, as far as differentiation with respect to $\mathbf{r}$ is concerned), and $\nabla^2\left(\frac{1}{\mathscr{r}}\right) = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$ (Eq. 1.102).

So $\nabla^2\mathbf{A} = \frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{r}')\left[-4\pi\delta^3(\boldsymbol{\mathscr{r}})\right]d\tau' = -\mu_0\mathbf{J}(\mathbf{r}).$ ✓