Solutions for sample problems

Note Title 2/3/2008

1) $\Psi(x,0) = \sqrt{\frac{8}{5a}}\ \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{2}{5a}}\ \sin\left(\frac{2\pi x}{a}\right)$

$\psi_n(x) = \sqrt{\frac{2}{a}}\ \sin(n\pi x/a)$

So $\Psi(x,0) = \underbrace{\sqrt{4/5}}_{c_1}\ \underbrace{\sqrt{\frac{2}{a}}\sin(\pi x/a)}_{\psi_1} + \underbrace{\sqrt{1/5}}_{c_2}\ \underbrace{\sqrt{\frac{2}{a}}\sin(\quad)}_{\psi_2}$.

$E_1 = \frac{\pi^2\hbar^2}{2ma^2} \qquad E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$

$$\Psi(x,t_0) = \sqrt{\frac{2}{5a}}\left\{2\sin\left(\frac{\pi x}{a}\right)e^{-i(\pi^2\hbar/2ma^2)t_0} + \sin\left(\frac{2\pi x}{a}\right)e^{-i(2\pi^2\hbar/ma^2)t_0}\right\}$$

$$\langle H\rangle = |c_1|^2 E_1 + |c_2|^2 E_2$$

$$= \frac{4}{5}\left(\frac{\pi^2\hbar^2}{2ma^2}\right) + \frac{1}{5}\left(\frac{4\pi^2\hbar^2}{2ma^2}\right) = \frac{4}{5}\frac{\pi^2\hbar^2}{ma^2}$$

This is independent of time

$$P\left(0 \le x \le \frac{a}{2}\right) = \int_0^{a/2} |\Psi(x, t_0)|^2 \, dx$$

$$\Psi(x,t) = C_1 \psi_1 e^{-iE_1 t_0/\hbar} + C_2 \psi_2 e^{-iE_2 t_0/\hbar}$$

$$|\Psi|^2 = \left(C_1 \psi_1 e^{+iE_1 t/\hbar} + C_2 \psi_2 e^{+iE_2 t_0/\hbar}\right)$$

$$\times \left(C_1 \psi_1 e^{-iE_1 t/\hbar} + C_2 \psi_2 e^{-iE_2 t_0/\hbar}\right)$$

$$= C_1^2 \psi_1^2 + C_2^2 \psi_2^2 + C_1 C_2 \psi_1 \psi_2 e^{-i(E_1-E_2)t_0/\hbar}$$

$$+ C_2 C_1 \psi_2 \psi_1 e^{+i(E_1-E_2)t_0/\hbar}$$

$$= C_1^2 \psi_1^2 + C_2^2 \psi_2^2 + C_1 C_2 \psi_1 \psi_2 \left[2\cos[(E_1 - E_2) t_0/\hbar \right]$$

$$|\Psi(x,t)|^2 = \frac{8}{5a} \sin^2\left(\frac{\pi x}{a}\right) + \frac{2}{5a} \sin^2\left(\frac{2\pi x}{a}\right) + \frac{8}{5a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \times$$

$$\cos\left[\frac{3\pi^2 \hbar}{2ma^2} t_0\right]$$

$$\int_0^{a/2} |\psi(x,t_0)|^2 \, dx = \frac{2}{5} + \frac{1}{10} + \frac{16}{15\pi} \cos\left(\frac{3\pi^2 \hbar \, t_0}{2ma^2}\right)$$

$$= \frac{1}{2} + \frac{16}{15\pi} \cos\left(\frac{3\pi^2 \hbar \, t_0}{2ma^2}\right)$$

2) $E_n = n^2 \times$ stuff that does not dep. on n.

So $E_2 = 4E_1$

$= 152 \text{ eV}$

I will give you these integrals on the exam: you do not need to memorize them for this exam

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4}\sin(2x)$$

You should know the following by heart

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$$

From this all else follows by substit.

Eg.

$$\int_{-\infty}^{\infty} e^{-x^2/a} \, dx = \sqrt{\pi a}$$

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$$\psi(x) = A\,(x/x_0)^2\, e^{-x/x_0}$$

$$\frac{d^2\psi}{dx^2} = A\, e^{-x/x_0}\, \frac{(x^2 - 4 x x_0 + 2 x_0^2)}{x_0^4}$$

$$= A\, e^{-x/x_0} \left[\frac{1}{x_0^2}\left(\frac{x}{x_0}\right)^2 - \frac{4}{x x_0}\left(\frac{x}{x_0}\right)^2 + 2\cancel{\left(\frac{x}{x_0}\right)^2} \frac{1}{x^2} \right]$$

$$= \psi(x) \left[\frac{2}{x^2} - \frac{4}{x x_0} + \frac{1}{x_0^2} \right]$$

$$\Rightarrow \quad = (E - V)\,\psi(x) = -\frac{\hbar^2}{2m}\,\psi(x) \left[\qquad \right]$$

$$\Rightarrow \quad E - V = -\frac{\hbar^2}{2m} \left(\frac{2}{x^2} - \frac{4}{x x_0} + \frac{1}{x_0^2} \right)$$

$$\lim_{x \to \infty} V(x) = 0 \quad \Rightarrow \quad \boxed{E = -\frac{\hbar^2}{2m}\,\frac{1}{x_0^2}}$$

$$\Rightarrow \quad \boxed{V = \frac{\hbar^2}{2m} \left(\frac{2}{x^2} - \frac{4}{x_0 x} \right)}$$

④ $\psi(x, t_0) = A\psi_0 + B\psi_1$

$$|\psi(x,t_0)|^2 = (A\psi_0 + B\psi_1)^2$$

so $\int |\psi(x,t_0)|^2 dx = A^2 + B^2 = 1$

$$\langle x \rangle = \int x \, (A\psi_0 + B\psi_1)^2$$

$$= \int x \left[A^2 \psi_0^2 + B^2 \psi_1^2 + 2AB\,\psi_0\psi_1 \right]$$

$$\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-x^2/2}$$

$$\psi_1(x) = \sqrt{2}\left(\frac{\alpha}{\pi}\right)^{1/4} x\, e^{-x^2/2} \qquad \alpha = \frac{m\omega}{\hbar}$$

$$y^2 = \alpha x$$

$$\int x \underbrace{e^{-x^2}}_{\psi_0^2} dx = 0$$

$$\int x \cdot \underbrace{x^2 e^{-x^2}}_{\psi_1^2} dx = 0$$

NB we don't have to explicitly evaluate these integ. to know they are zero.

$$\text{so} \quad \langle x \rangle = 2AB \int_{-\infty}^{\infty} x \underbrace{e^{-x^2/2}}_{\psi_1} \cdot \underbrace{x e^{-x^2/2}}_{\psi_2} dx$$

$$\neq 0$$

5) $$a^{+}a = \frac{m\omega}{2\hbar}\left(x - \frac{i}{m\omega}p\right)\left(x + \frac{i}{m\omega}p\right)$$

$$= \frac{m\omega}{2\hbar}\left[x^2 + \frac{i}{m\omega}\underbrace{(xp - px)}_{i\hbar} + \frac{p^2}{m^2\omega^2}\right]$$

$$= \frac{m\omega}{2\hbar}\left[x^2 - \frac{\hbar}{m\omega} + \frac{p^2}{m^2\omega^2}\right]$$

$$= \frac{m\omega}{2\hbar}x^2 + \frac{p^2}{2\hbar m\omega} - \frac{1}{2}$$

$$= \frac{1}{\hbar\omega}\left[\frac{1}{2}m\omega^2 + \frac{p^2}{2m}\right] - \frac{1}{2}$$

$$= \frac{1}{\hbar\omega}H - 1/2$$

6) $$[x, p]\,\phi = x\left(-i\hbar\frac{d}{dx}\right)\phi - \left(-i\hbar\frac{d}{dx}\right)x\,\phi$$

$\uparrow$ arb. function

$$= \underbrace{-i\hbar\, x\phi' + i\hbar\, x\phi'}_{0} + i\hbar\,\phi$$

$$[x, p]\,\phi = i\hbar\,\phi \Rightarrow [x, p] = i\hbar$$

b) $\hbar\omega\left[n + \frac{1}{2}\right]$

c) $$\int_{-\infty}^{\infty} \frac{d^2\psi^*}{dx^2}\psi - \psi^*\frac{d^2\psi}{dx^2}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{dx}\left[\frac{d\psi^*}{dx}\psi - \psi^*\frac{d\psi}{dx}\right]dx$$

$$= \left[\frac{d\psi^*}{dx}\psi - \psi^*\frac{d\psi}{dx}\right]_{-\infty}^{\infty}$$

$= 0$ since $\psi, \psi' \to 0$ as $x \to \infty$

7. A stationary state is an E-state of the Hamiltonian. It is called stationary since the probability desity has no time dependence

$$\Psi_n(x,t) = \psi_n(x)\, e^{-iE_n t/\hbar}$$

$$|\Psi_n|^2 = |\psi_n(x)|^2$$

if a system is initialized in such a state it wavefunction for later time is

$$\Psi(x,t) = \psi_n(x)\, e^{-iE_n t/\hbar}$$

So it never leaves this state.

a) de Broglie wave is a QM plane wave state

$$\psi(x,t) = e^{i(p\cdot x - Et)/\hbar}$$

it is associated with a perfectly well defined momentum and hence has infinite spatial extent

$$\langle p \rangle = \int_{-\infty}^{\infty} e^{-i(p\cdot x)/\hbar} \left(-i\hbar \frac{d}{dx}\right) e^{i(p\cdot x)/\hbar}\, dx$$

$$= \int_{-\infty}^{\infty} p \, dx = p \int_{-\infty}^{\infty} dx = \infty$$

so instead look at $p \int_{-A/2}^{A/2} dx$

$$= \frac{p}{A}$$

Similarly $\langle p^2 \rangle = p^2 \int_{-A/2}^{A/2} dx = \frac{p^2}{A}$

So $\sigma_p = \langle p^2 \rangle - \langle p \rangle^2 = 0$

9) $$\Psi(x,t) = A e^{-a[mx^2/\hbar + it]}$$

$$\int_{-\infty}^{\infty} |\Psi|^2 \, dx = A^2 \sqrt{\frac{\hbar \pi}{2ma}} = 1 \Rightarrow$$

$$A = \left(\frac{2ma}{\hbar \pi}\right)^{1/4}$$

$$\langle x \rangle = \sqrt{\frac{2ma}{\hbar \pi}} \int_{-\infty}^{\infty} x \, e^{-2am/\hbar \, x^2} \, dx = 0$$

$$\langle x^2 \rangle = \sqrt{} \int x^2 e^{-2ma/\hbar \, x^2} dx = \frac{\hbar}{4am}$$

$$\langle p \rangle = 0$$

$$\langle p^2 \rangle = a \hbar m$$

$$\sigma_x = \sqrt{\frac{\hbar}{4am}}$$

$$\sigma_p = \sqrt{a \hbar m}$$

$$\boxed{\sigma_x \sigma_p = \frac{\hbar}{2}}$$