Problem 3.18

$$V_0(\theta) = k\cos(3\theta) = k\left[4\cos^3\theta - 3\cos\theta\right] = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right].$$

(I know that any 3rd order polynomial can be expressed as a linear combination of the first four Legendre polynomials; in this case, since the polynomial is *odd*, I only need P_1 and P_3.)

$$4\cos^3\theta - 3\cos\theta = \alpha\left[\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] + \beta\cos\theta = \frac{5\alpha}{2}\cos^3\theta + \left(\beta - \frac{3}{2}\alpha\right)\cos\theta,$$

so

$$4 = \frac{5\alpha}{2} \Rightarrow \alpha = \frac{8}{5}; \quad -3 = \beta - \frac{3}{2}\alpha = \beta - \frac{3}{2}\cdot\frac{8}{5} = \beta - \frac{12}{5} \Rightarrow \beta = \frac{12}{5} - 3 = -\frac{3}{5}.$$

Therefore

$$V_0(\theta) = \frac{k}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right].$$

Now

$$V(r,\theta) = \left\{\begin{array}{ll} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & \text{for } r \le R \quad \text{(Eq. 3.66)} \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & \text{for } r \ge R \quad \text{(Eq. 3.71)} \end{array}\right\},$$

where

$$\begin{aligned} A_l &= \frac{(2l+1)}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\,d\theta \quad \text{(Eq. 3.69)} \\ &= \frac{(2l+1)}{2R^l}\frac{k}{5}\left\{8\int_0^\pi P_3(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta - 3\int_0^\pi P_1(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta\right\} \\ &= \frac{k}{5}\frac{(2l+1)}{2R^l}\left\{8\frac{2}{(2l+1)}\delta_{l3} - 3\frac{2}{(2l+1)}\delta_{l1}\right\} = \frac{k}{5}\frac{1}{R^l}\left[8\,\delta_{l3} - 3\,\delta_{l1}\right] \\ &= \left\{\begin{array}{ll} 8k/5R^3, & \text{if } l = 3 \\ -3k/5R, & \text{if } l = 1 \end{array}\right\} \text{(zero otherwise).} \end{aligned}$$

Therefore

$$V(r,\theta) = -\frac{3k}{5R}rP_1(\cos\theta) + \frac{8k}{5R^3}r^3P_3(\cos\theta) = \boxed{\frac{k}{5}\left[8\left(\frac{r}{R}\right)^3 P_3(\cos\theta) - 3\left(\frac{r}{R}\right)P_1(\cos\theta)\right]},$$

or

$$\frac{k}{5}\left\{8\left(\frac{r}{R}\right)^3\frac{1}{2}\left[5\cos^3\theta - 3\cos\theta\right] - 3\left(\frac{r}{R}\right)\cos\theta\right\} \Rightarrow \boxed{V(r,\theta) = \frac{k}{5}\frac{r}{R}\cos\theta\left\{4\left(\frac{r}{R}\right)^2\left[5\cos^2\theta - 3\right] - 3\right\}}$$

(for $r \le R$). Meanwhile, $B_l = A_l R^{2l+1}$ (Eq. 3.81—this follows from the continuity of V at R). Therefore

$$B_l = \left\{\begin{array}{ll} 8kR^4/5, & \text{if } l = 3 \\ -3kR^2/5, & \text{if } l = 1 \end{array}\right\} \quad \text{(zero otherwise).}$$

So

$$V(r,\theta) = \frac{-3kR^2}{5}\frac{1}{r^2}P_1(\cos\theta) + \frac{8kR^4}{5}\frac{1}{r^4}P_3(\cos\theta) = \boxed{\frac{k}{5}\left[8\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - 3\left(\frac{R}{r}\right)^2 P_1(\cos\theta)\right]},$$

or

$$\boxed{V(r,\theta) = \frac{k}{5}\left(\frac{R}{r}\right)^2\cos\theta\left\{4\left(\frac{R}{r}\right)^2\left[5\cos^2\theta - 3\right] - 3\right\}}$$

(for $r \ge R$). Finally, using Eq. 3.83:

$$\begin{aligned} \sigma(\theta) &= \epsilon_0\sum_{l=0}^{\infty}(2l+1)A_lR^{l-1}P_l(\cos\theta) = \epsilon_0\left[3A_1P_1 + 7A_3R^2P_3\right] \\ &= \epsilon_0\left[3\left(-\frac{3k}{5R}\right)P_1 + 7\left(\frac{8k}{5R^3}\right)R^2P_3\right] = \boxed{\frac{\epsilon_0 k}{5R}\left[-9P_1(\cos\theta) + 56P_3(\cos\theta)\right]} \\ &= \frac{\epsilon_0 k}{5R}\left[-9\cos\theta + \frac{56}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] = \frac{\epsilon_0 k}{5R}\cos\theta\left[-9 + 28\cdot 5\cos^2\theta - 28\cdot 3\right] \\ &= \boxed{\frac{\epsilon_0 k}{5R}\cos\theta\left[140\cos^2\theta - 93\right]}. \end{aligned}$$