Resonators and stability

Examples of resonators

Unfolded resonator and ABCD description

Stability

- Ray picture
- Gaussian beam picture

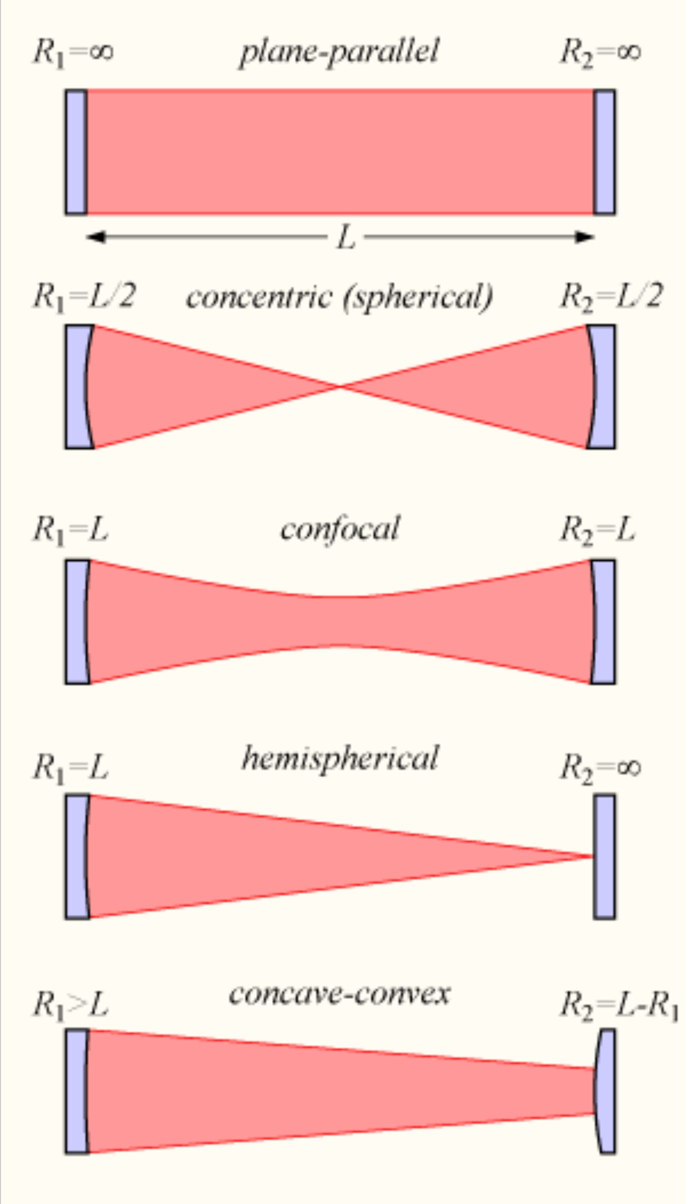
2 mirror resonators and the stability map

Analysis of resonators

- beam sizes

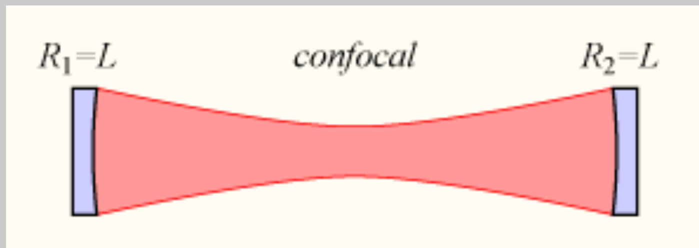
Resonators

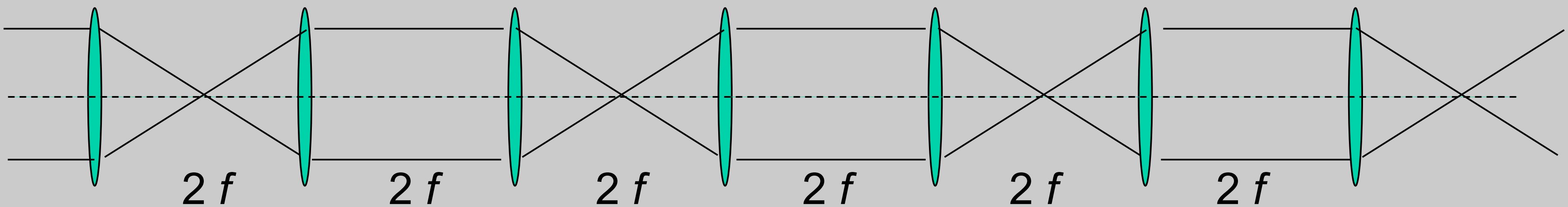
- Resonators provide feedback for the photons to build up by passing through the gain medium
- Curved mirrors are typically used to control the beam size inside the gain medium
- Types of resonators
  - Many resonators have more than two mirrors, but most can be mapped onto a two-mirror system.



Periodic lens model

- A resonator can be “unfolded” by modeling the curved mirrors as ideal lenses

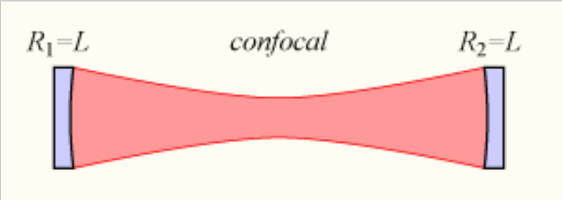




- Are there rays that will stay confined?
- If so, resonator is ***stable***.

Resonator ABCD model

- Build a ABCD matrix model of the periodic lens sequence
  - First get a *round-trip* matrix



$$M_{RT}(f,L)=\begin{pmatrix} A & B \\ C & D \end{pmatrix}=M_L(f)\cdot M_T(L)\cdot M_L(f)\cdot M_T(L)$$

$$=\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

  - Free to choose starting point
  - Focal length ***f*** and mirror separation ***L*** can vary

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} r_0 \\ r_0' \end{pmatrix}$$

After 2 round trips

Resonator stability: ray picture

- Will a ray stay trapped?
- Look at whether r_n and r'_n stay finite as n goes to infinity

$$\begin{pmatrix} r_n \\ r'_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \begin{pmatrix} r_0 \\ r'_0 \end{pmatrix}$$

- Method:
  - Diagonalize matrix:

$$M_{RT} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1}$$

  - then

$$M_{RT}{}^n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1} U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1} \cdots$$

$$= U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix}^n U^{-1} = U \begin{pmatrix} \lambda_a{}^n & 0 \\ 0 & \lambda_b{}^n \end{pmatrix} U^{-1}$$

Stability condition

- The ray will stay trapped (stable resonator) if

$$|\lambda_a| \le 1 \quad |\lambda_b| \le 1 \qquad \frac{1}{|\lambda_a|} \le 1 \quad \frac{1}{|\lambda_b|} \le 1$$

Reverse propagation

- Therefore matrix eigenvalues must satisfy $|\lambda_a| = |\lambda_b| = 1$
- Property of ABCD: $\det M_{RT} = \lambda_a \lambda_b = 1$

$$\therefore \lambda_a = e^{i\theta}, \lambda_b = \lambda_a^*$$

- Trace of M is invariant upon rotation of matrix:

$$\mathrm{Tr}\, M_{RT} = \lambda_a + \lambda_b = A + D$$

$$= e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

- Finally stability condition is: $-1 \le \frac{A+D}{2} \le 1$

Some properties of ABCD matrices

1. Determinant = 1 if start and end points are in the same medium (same refr. Index)
   – Example: $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$
   – Counter example: dielectric interface $\begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$
   – Therefore, det M = 1, but note that eigenvalues can be real or complex
2. Complex eigenvalues are of the form $e^{\pm i\theta}$
   – Outside of stability range, eigenvalues are real

   $$\lambda_a = 1/\lambda_b \qquad \mathrm{Tr}\, M_{RT} = \lambda_a + 1/\lambda_a > 2 \quad \text{if } \lambda_a > 1$$
3. M is not necessarily unitary (where $M^{-1} = M^{\dagger}$)

Stability for Gaussian beams in resonators

- A stable resonator mode is one that repeats itself on each round trip
  - Amplitude and phase are matched $\therefore q_{n+1} = q_n$

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} = q_0 \rightarrow Aq_0 + B = q_0\left(Cq_0 + D\right)$$

$$\rightarrow 0 = Cq_0{}^2 + \left(D - A\right)q_0 - B$$

$$q_0 = \frac{\left(A - D\right)}{2C} \pm \frac{1}{2C}\sqrt{\left(A - D\right)^2 + 4BC}$$

  - Since $\frac{1}{q_0} = \frac{1}{R} - i\frac{\lambda}{\pi w^2}$ q_0 must be complex (w is finite)

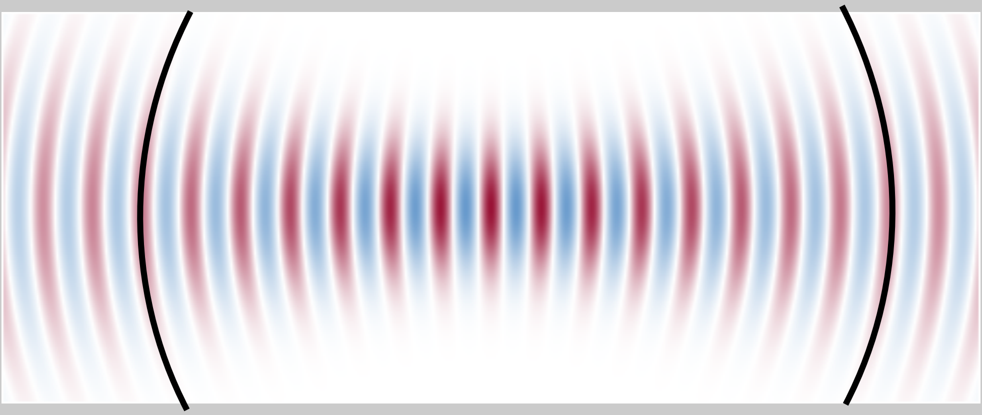
$$\therefore \left(A - D\right)^2 + 4BC < 0$$

Stability for Gaussian beams in resonators

- We know: $(A-D)^2 + 4BC < 0$
- And, since det(M) = 1 $\quad AD - BC = 1$

$$(A-D)^2 + 4BC = (A-D)^2 + 4(AD-1)$$
$$= A^2 - 2AD + D^2 + 4AD - 4$$
$$= (A+D)^2 - 4 < 0$$

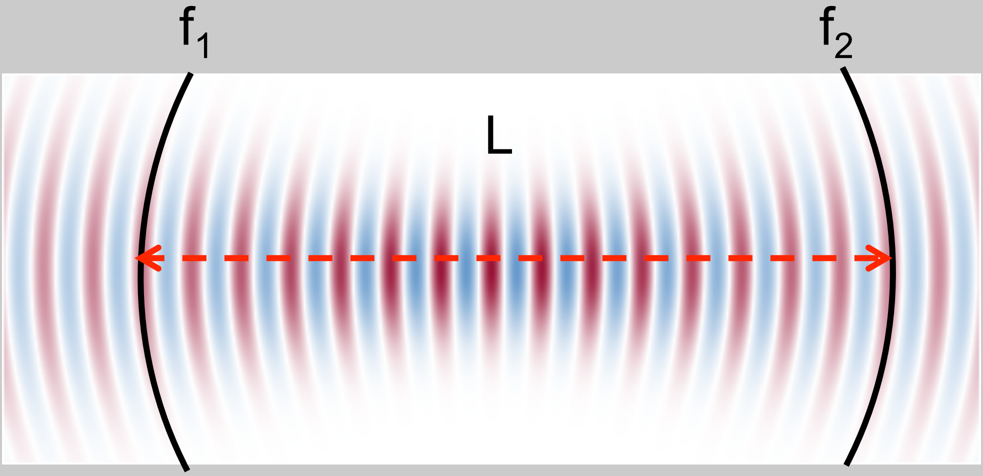
- Stability condition: $\dfrac{(A+D)^2}{4} < 1$



If this condition is satisfied, curvature of each end mirror matches wavefront curvature.

2 mirror cavity stability

- Important example
  - many resonators can be mapped to a 2 mirror cavity



$$M = \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & L \\ -1/f_1 & 1 - L/f_1 \end{pmatrix} \begin{pmatrix} 1 & L \\ -1/f_2 & 1 - L/f_2 \end{pmatrix}$$

Stability for 2 mirror resonator

- Stability condition: $\frac{(A+D)^2}{4} < 1 \rightarrow -1 < \frac{A+D}{2} < 1$
  - Evaluate A and D from round-trip matrix

$$M = \begin{pmatrix} 1 & L \\ -1/f_1 & 1-L/f_1 \end{pmatrix} \begin{pmatrix} 1 & L \\ -1/f_2 & 1-L/f_2 \end{pmatrix}$$

$$A = 1 - L/f_2$$

$$D = -L/f_1 + (1-L/f_1)(1-L/f_2)$$

$$f_1 = R_1/2 \qquad f_2 = R_2/2$$

$$\frac{A+D}{2} = \frac{1}{2}\left(1 - \frac{2L}{R_2} - \frac{2L}{R_1} + 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{4L^2}{R_1 R_2}\right)$$

$$= 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{2L^2}{R_1 R_2} = 2\left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) - 1 \equiv 2 g_1 g_2 - 1$$

2 mirror stability and the stability map

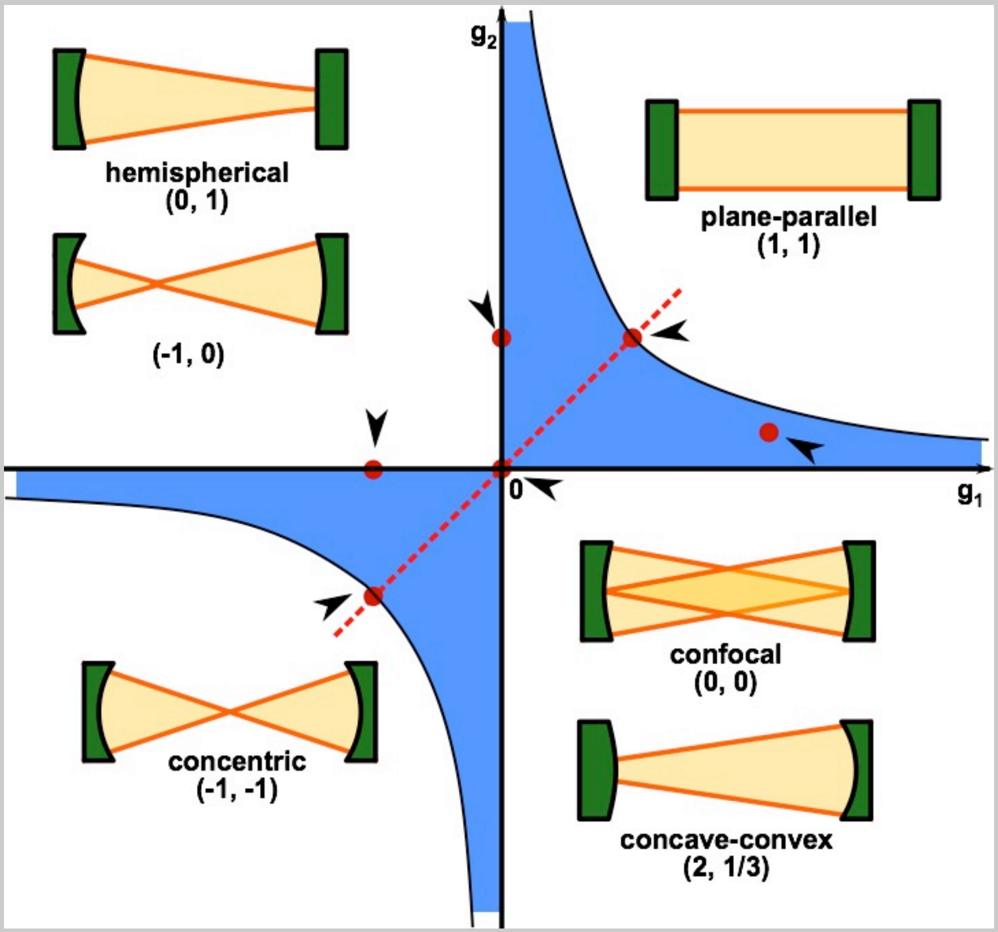
- Cavity is stable if $-1 < \frac{A+D}{2} < 1$

$$-1 < 2g_1 g_2 - 1 < 1$$

Stable in shaded regions
Unstable in white regions

$$0 \le g_1 g_2 \le 1$$

$$g_1 = 1 - \frac{L}{R_1} \qquad g_2 = 1 - \frac{L}{R_2}$$



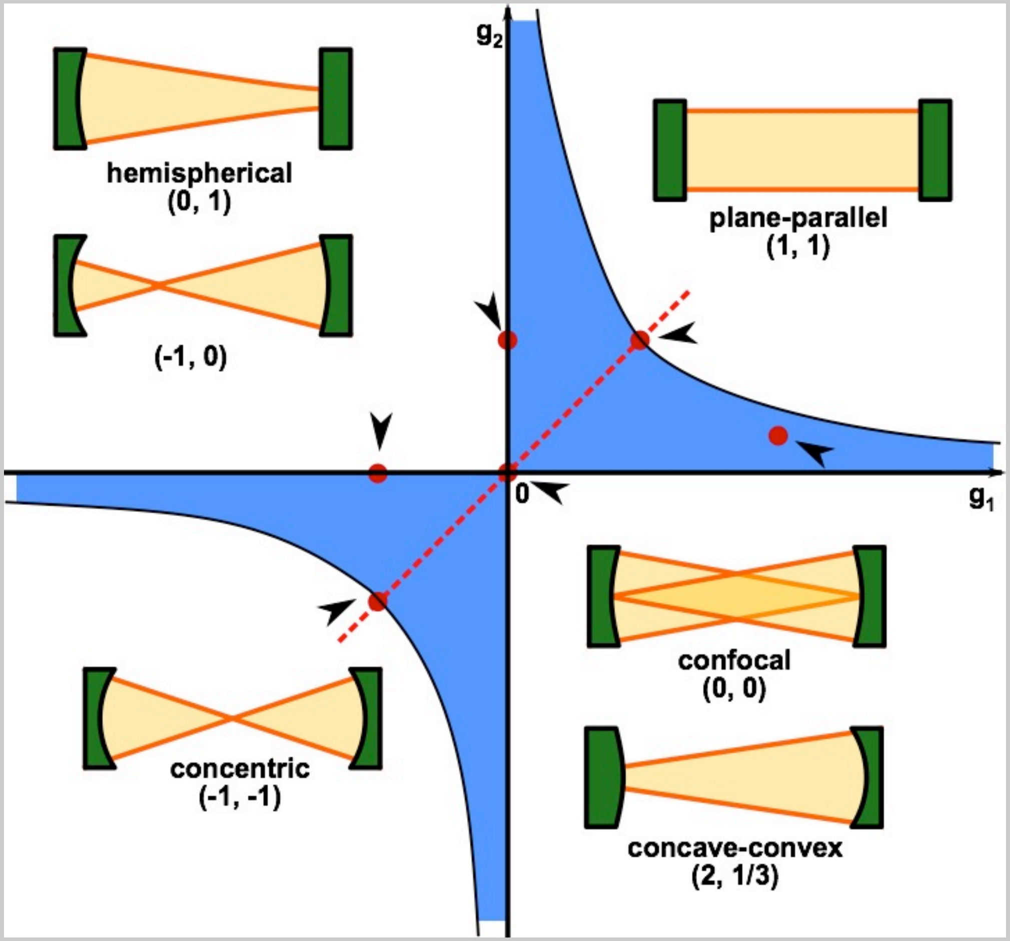
1st and 3rd quadrants:
Positive branch:
$0 < g_1 g_2 < 1$ **stable**
$g_1 g_2 > 1$ **unstable**
No focal point inside resonator

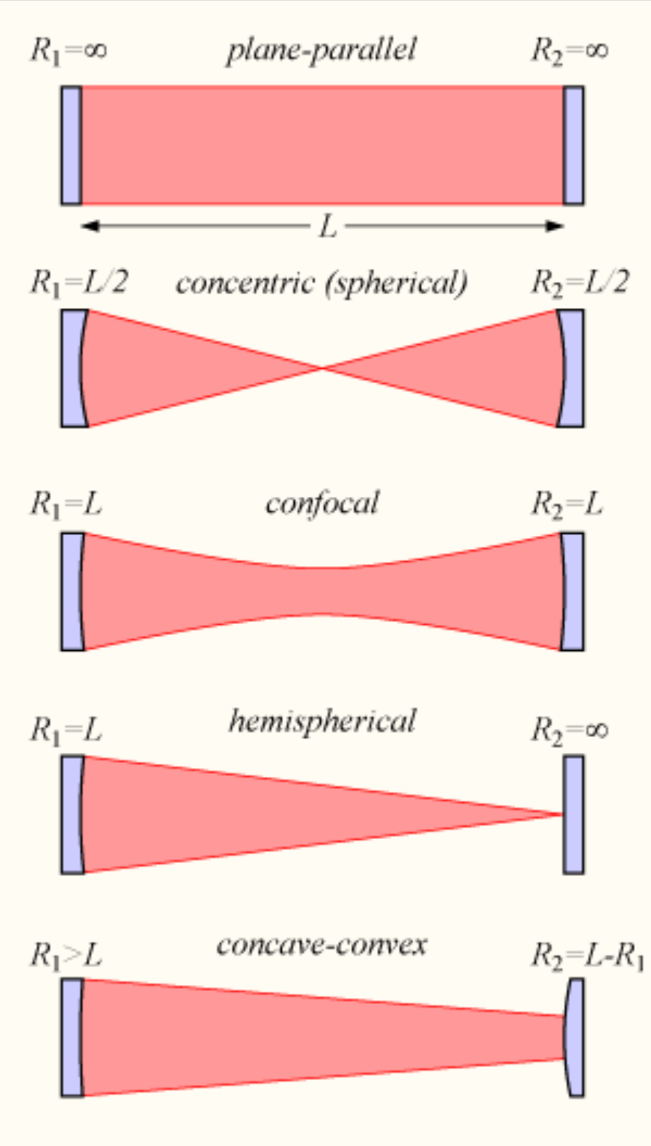
2nd and 4th quadrants:
Negative branch: $g_1 g_2 < 0$
One center of curvature inside resonator
focal point inside resonator

Boundaries of stability

$$g_1 = 1 - \frac{L}{R_1} \quad g_2 = 1 - \frac{L}{R_2}$$

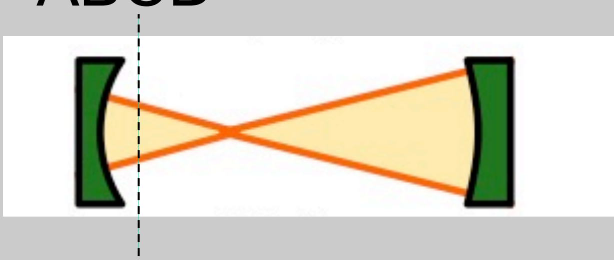
- Easily identified stable resonators are actually at edge of stabilty



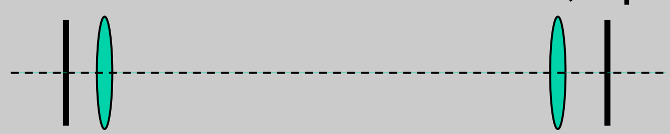


Determining beam sizes

- From *q* parameter
  - For stable mode: $$q_0 = \frac{(A-D)}{2C} \pm \frac{1}{2C}\sqrt{(A-D)^2 + 4BC}$$
  - And $$\frac{1}{q_0} = \frac{1}{R} - i\frac{\lambda}{\pi w^2}$$ Beam waist is where $\mathrm{Re}[1/q_0]=0$
  - So $$w^2 = -\frac{\lambda}{\pi \,\mathrm{Im}\left[q_0^{-1}\right]}$$
  - Which *w* is this? It is at the start/end position of the ABCD



For curved end mirror, split:



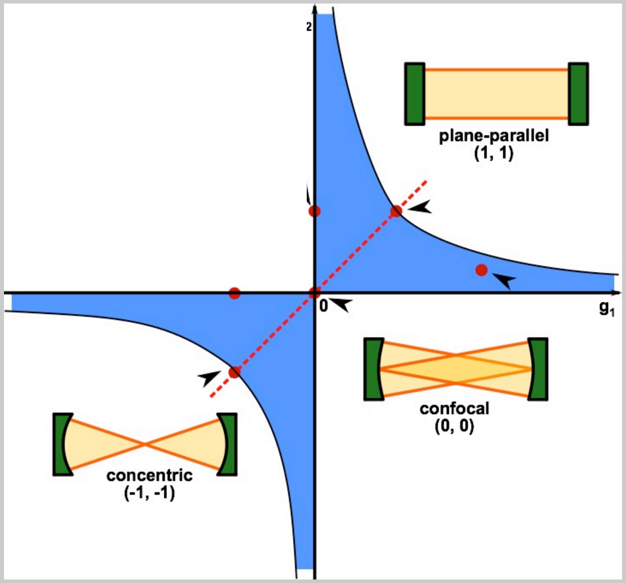
Then mode is collimated at end.

Symmetric cavities

- At end mirror, wavefront curvature matches surface of mirror.
  - Plano end mirror: waist at mirror
  - Symmetric cavity (R_1=R_2, g_1=g_2): waist location at center. Can fully specify mode w/o ABCD.
- Use Gaussian beam equations:

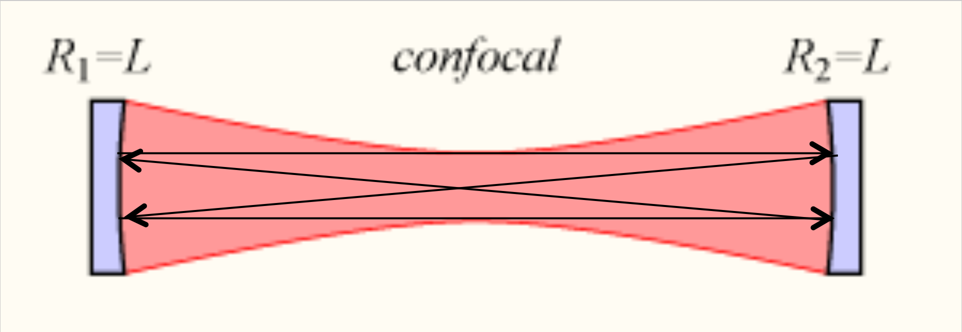
$$R = z\left(1+\frac{z_R^2}{z^2}\right) \rightarrow \frac{L}{2}\left(1+\frac{4z_R^2}{L^2}\right)$$

$$z_R = \frac{L}{2}\sqrt{\frac{2R}{L}-1} \qquad w_0 = \sqrt{\frac{\lambda L}{2\pi}\sqrt{\frac{2R}{L}-1}}$$



Confocal cavity

- Symmetric cavity, focal points overlap



  - Cavity length is equal to the confocal parameter
  - Spot size:

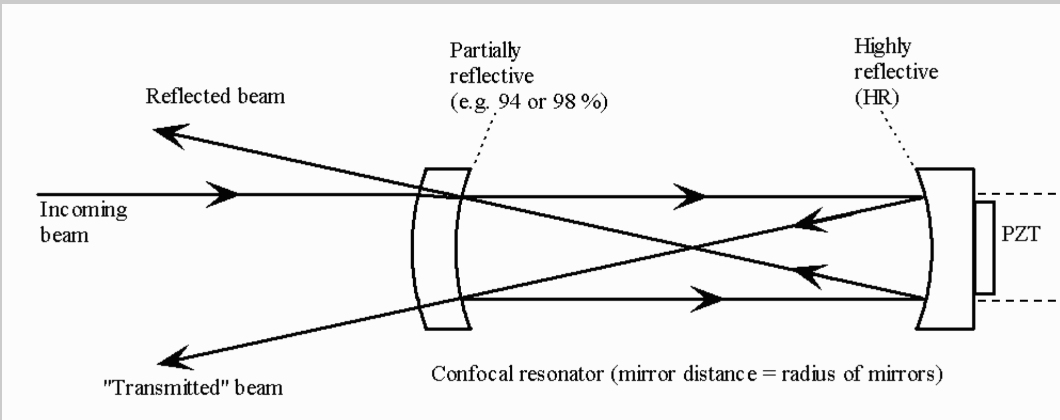
$$L = 2z_R = b$$

$$w_0 = \sqrt{\frac{\lambda L}{2\pi}}$$

  - Confocal cavity has only ~40% variation of mode size along cavity
  - Least sensitivity to angular misalignment.

Scanning Fabry-Perot interferometer

- Confocal resonator



Confocal Scanning Fabry-Perot Interferometer
(Drawing by Christoph Bollig)

Transmitted beams

Look for beam overlap

See fringes: transmission through curved mirrors makes beams diverge

- Mode-matching: make input beam identical to desired output beam
  - Set initial beam size and focusing lens

Example: 2GHz FP

- Free spectral range = 2GHz

$$\Delta\nu = \frac{c}{2L} \rightarrow L = \frac{c}{2\Delta\nu}$$

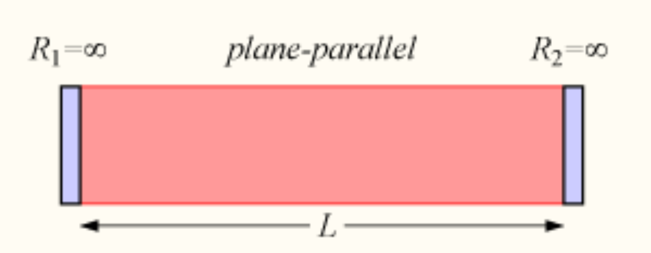
  - Cavity length L = 7.5cm
  - Mode waist radius:

$$w_0 = \sqrt{\frac{\lambda L}{2\pi}}$$

    w_0 ~ 87um (for 632.8nm)
  - Output mode waist radius: $\sqrt{2}w_0 = 123\text{um}$
  - In general, resonant frequency is different for higher-order modes. If confocal FP is well-aligned, all even modes are degenerate, and odd modes are midway between TEM00 mode frequencies.

Near-planar and concentric limits

- Near-planar: R very large, >> L



$$z_R = \frac{L}{2}\sqrt{\frac{2R}{L}-1} = \frac{L}{2}\frac{2R}{L}\sqrt{1-\frac{L}{2R}} \approx R\left(1-\frac{L}{4R}\right)$$

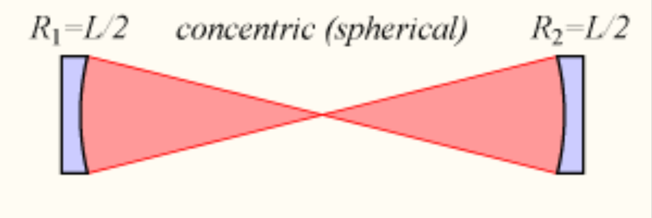
  – Large, constant mode size. sensitive to angle misalignment

- Near-concentric: L ~ 2R
  – Let L = 2R – δL

$$z_R = \frac{L}{2}\sqrt{\frac{2R}{L}-1} = \frac{2R-\delta L}{2}\sqrt{\frac{2R}{2R-\delta L}-1} \approx R\sqrt{\left(1+\frac{\delta L}{2R}\right)-1} \approx \sqrt{\frac{R\delta L}{2}}$$

  – Small mode in center, large mode at curved mirrors

In general, position on stability map controls mode size throughout cavity.



Higher-order resonator modes

– Higher-order resonator modes follow the Hermite-Gaussian (or Laguerre-Gaussian) funcitons

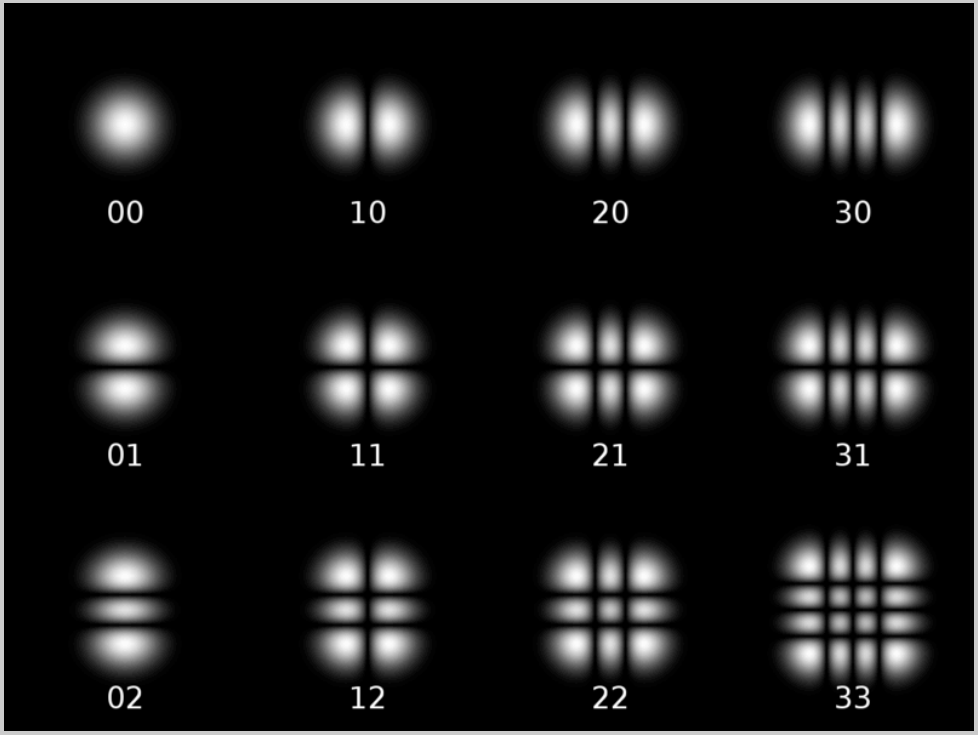
$$E(x,y,z) = A_0 e^{-i(kz-\eta_{lm}(z))} \frac{w_0}{w(z)} e^{-\frac{x^2+y^2}{w^2(z)}} H_l\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) e^{-i\frac{k(x^2+y^2)}{2R(z)}}$$

$$\eta_{lm} = (1+l+m)\tan^{-1}\left(\frac{z}{z_R}\right)$$

R(z) is independent of mode order

Resonant frequencies depend on mode indices.

Extent of field is larger as mode index increases – more diffraction loss.



Eigenvalues for high-order standing waves

- High-order modes generally have different resonant frequencies

$$\nu_{nlm} = \frac{c}{2L}\left(n + \left(\frac{1+l+m}{\pi} \right) \cos^{-1}\left(\pm\sqrt{AD} \right) \right)$$

  - 2 mirror resonator:

$$\nu_{nlm} = \frac{c}{2L}\left(n + \left(\frac{1+l+m}{\pi} \right) \cos^{-1}\left(\pm\sqrt{g_1 g_2} \right) \right)$$

+ if g_1 and $g_2 > 0$

- if g_1 and $g_2 < 0$

  - Confocal: $g_1 = g_2 = 0$

$$\nu_{nlm} = \frac{c}{4L}\left(2n + \left(1+l+m \right) \right)$$

Even modes are degenerate
Odd modes degenerate

Resonator stability analysis

- Resonators are designed under different constraints and can be optimized
- Plot a stability parameter to show stable zone(s) of operation
  - Stability condition: $-1 < \frac{A+D}{2} < 1$
  - By convention to plot *s* parameter:

$$s = 1 - \left(\frac{A+D}{2} \right)^2$$

Parameter is always positive in stable zone

Focusing resonator

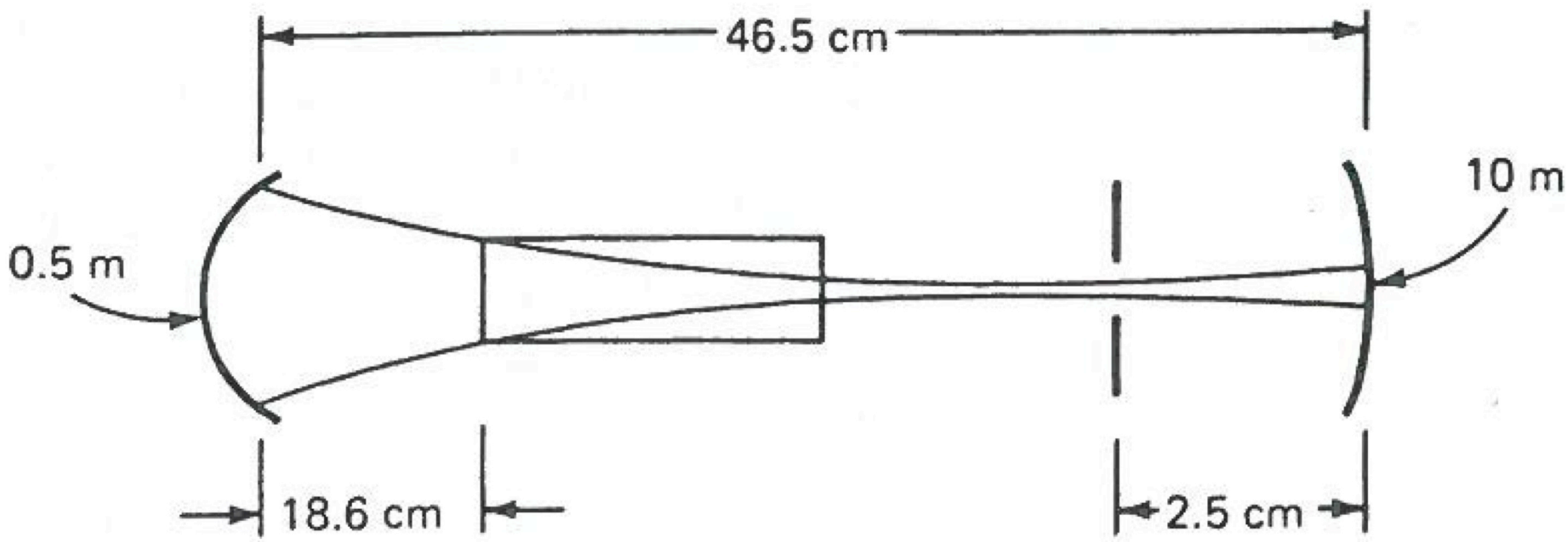
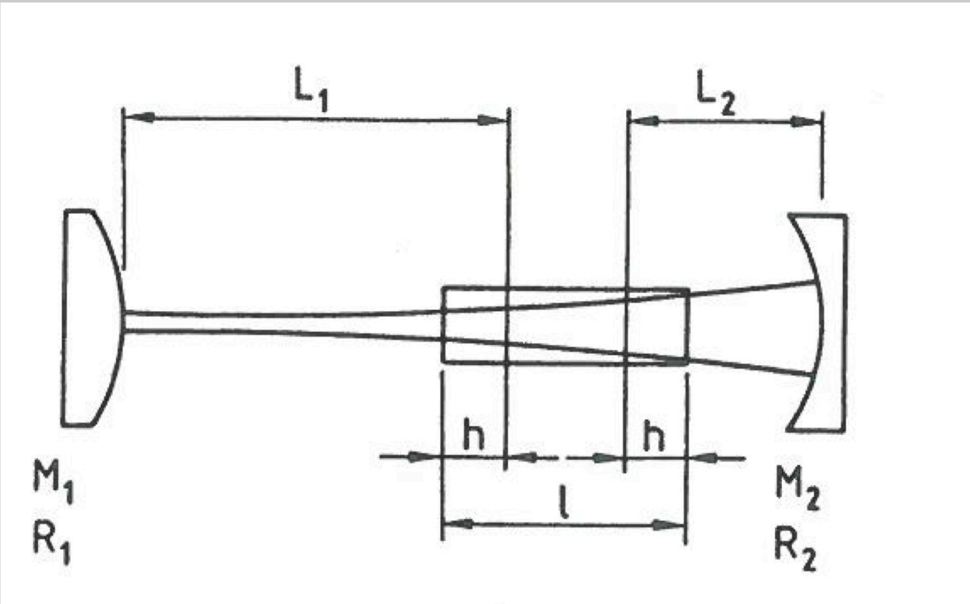


Fig. 5.20. Focusing resonator geometry

Nearly hemispherical resonator

- large mode on left
- Laser rod acts as aperture to limit TEM00 operation
- Second aperture to clean up beam

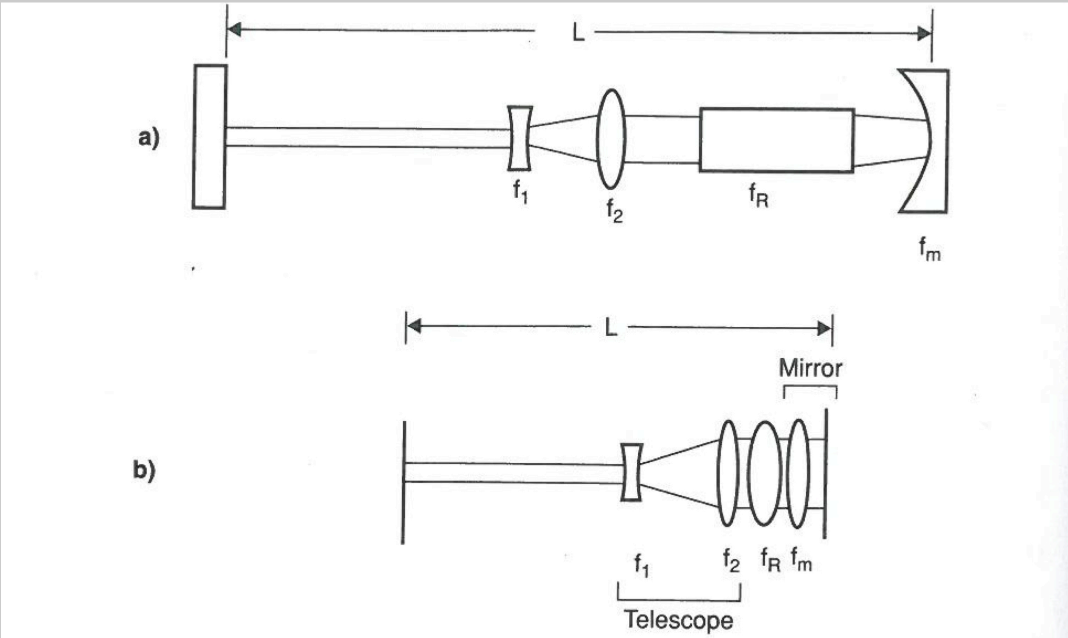
Convex-concave resonator

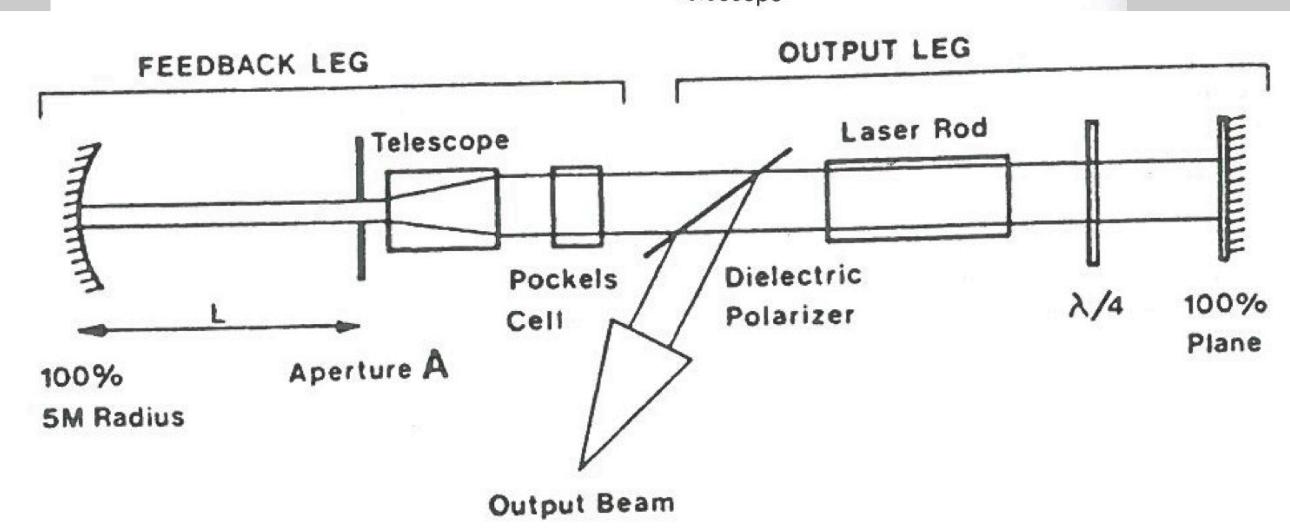


Weak thermal lensing in rod

- Small spot on convex mirror
- Too intense for pulsed operation

Internal telescope resonators





Astigmatic compensation

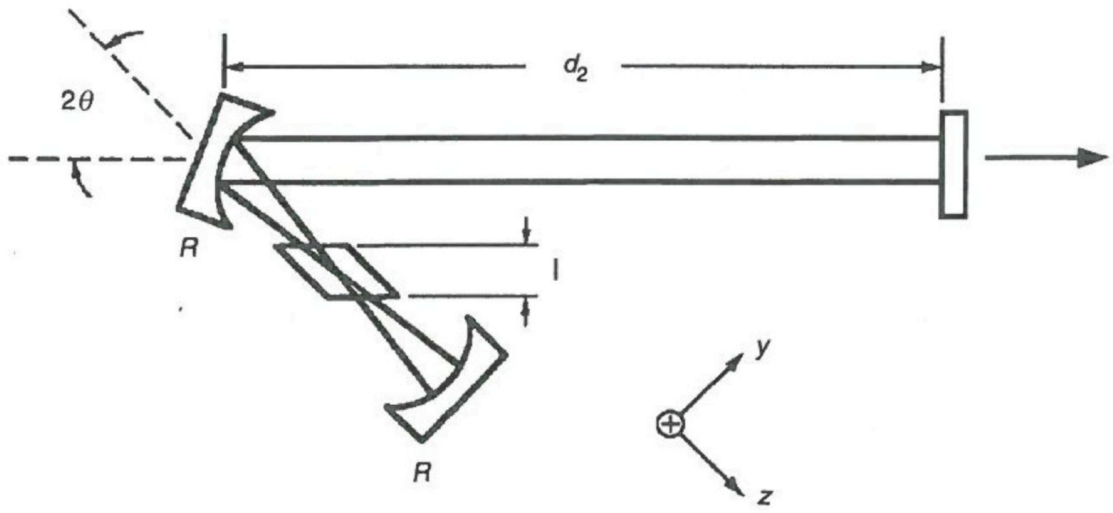


Fig. 5.29. Astigmatic compensation of a folded resonator containing an optical element at Brewster's angle

Mechanically-stable resonator design

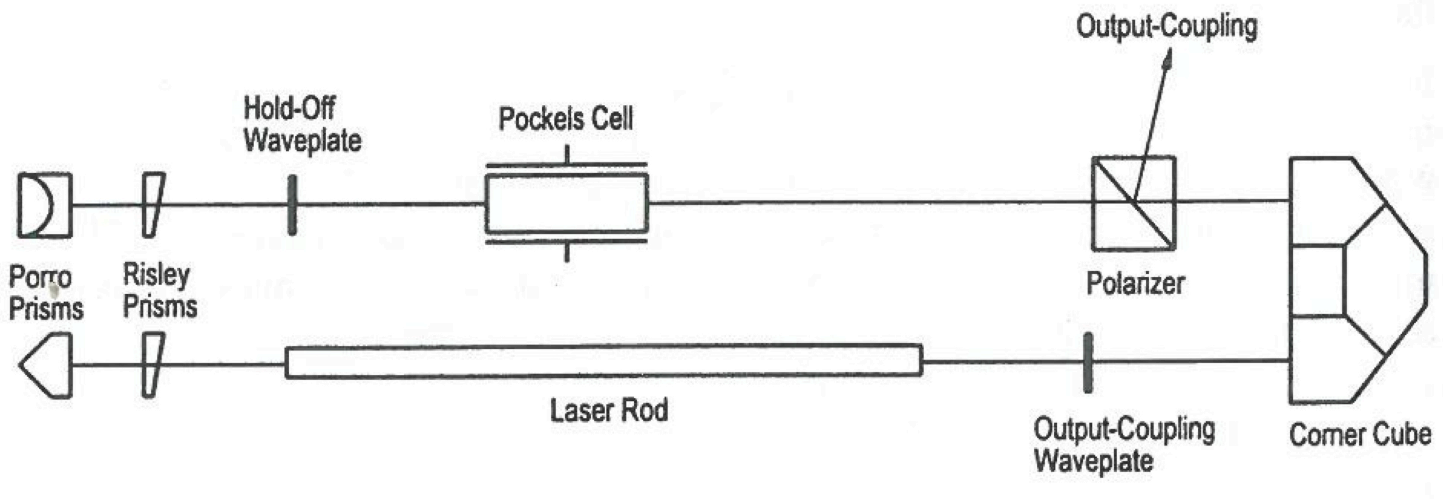


Fig. 5.30. Resonator using cross Porro prisms and polarization output coupling

2 mirror stability and the stability map

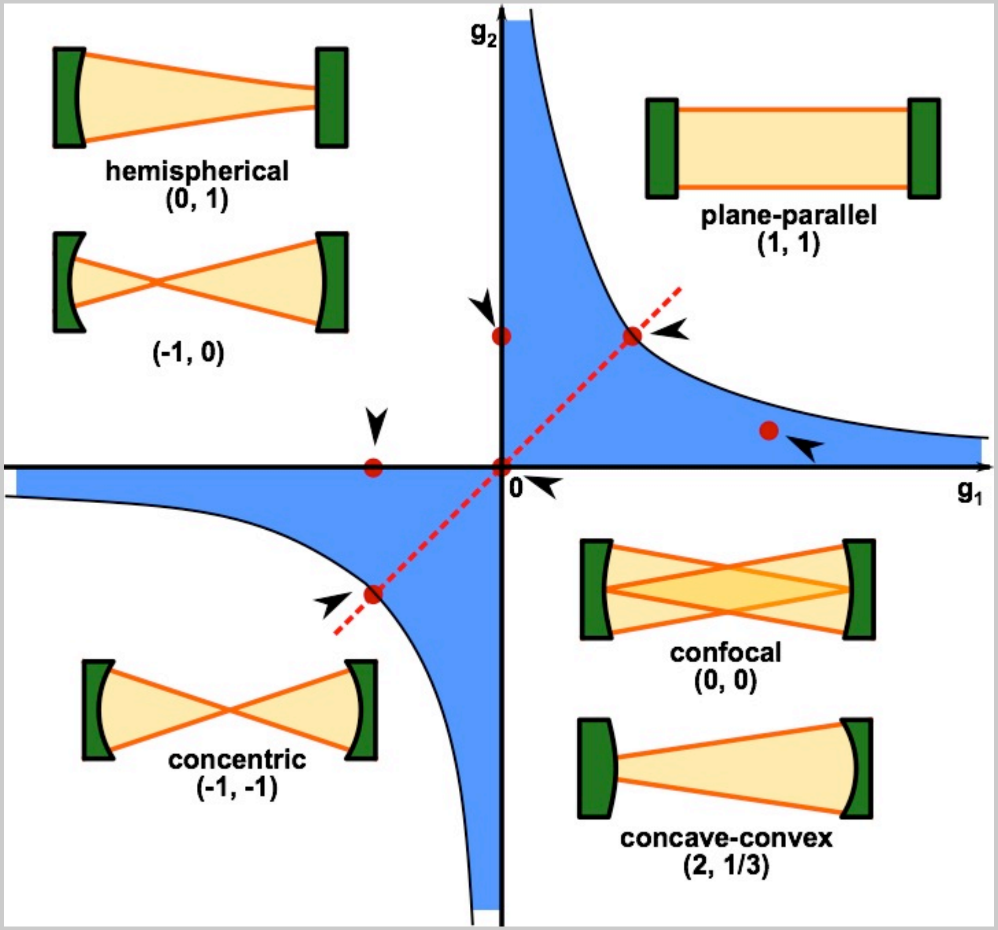
- Cavity is stable if $-1 < \frac{A+D}{2} < 1$

$$-1 < 2g_1 g_2 - 1 < 1$$

Stable in shaded regions
Unstable in white regions

$$0 \le g_1 g_2 \le 1$$

$$g_1 = 1 - \frac{L}{R_1} \qquad g_2 = 1 - \frac{L}{R_2}$$



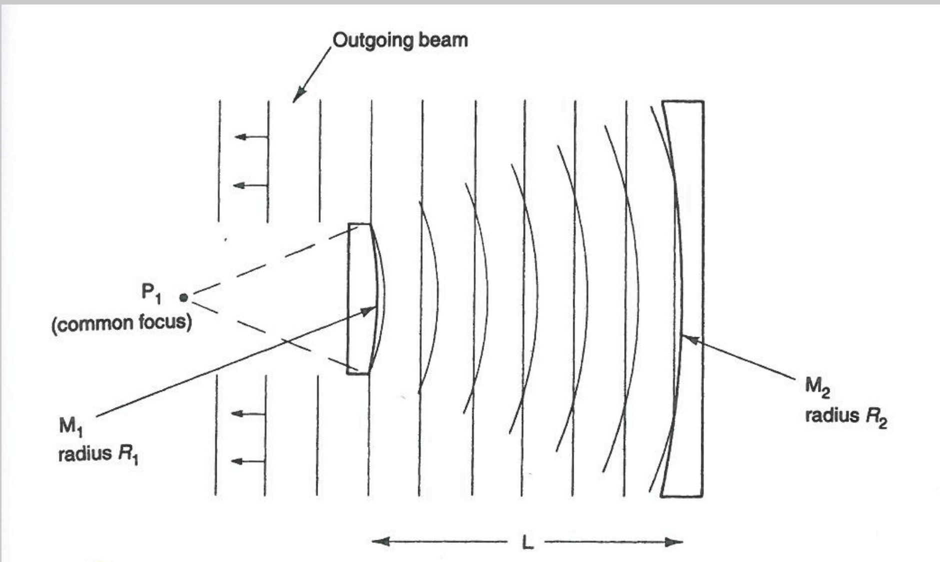
1st and 3rd quadrants:
Positive branch:
$0 < g_1 g_2 < 1$ **stable**
$g_1 g_2 > 1$ **unstable**
No focal point inside resonator

2nd and 4th quadrants:
Negative branch: $g_1 g_2 < 0$
One center of curvature inside resonator
focal point inside resonator

Unstable resonators

Unstable resonators often use beam magnification to output couple past a mirror.

- Gain must be sufficient to overcome diffractive losses.

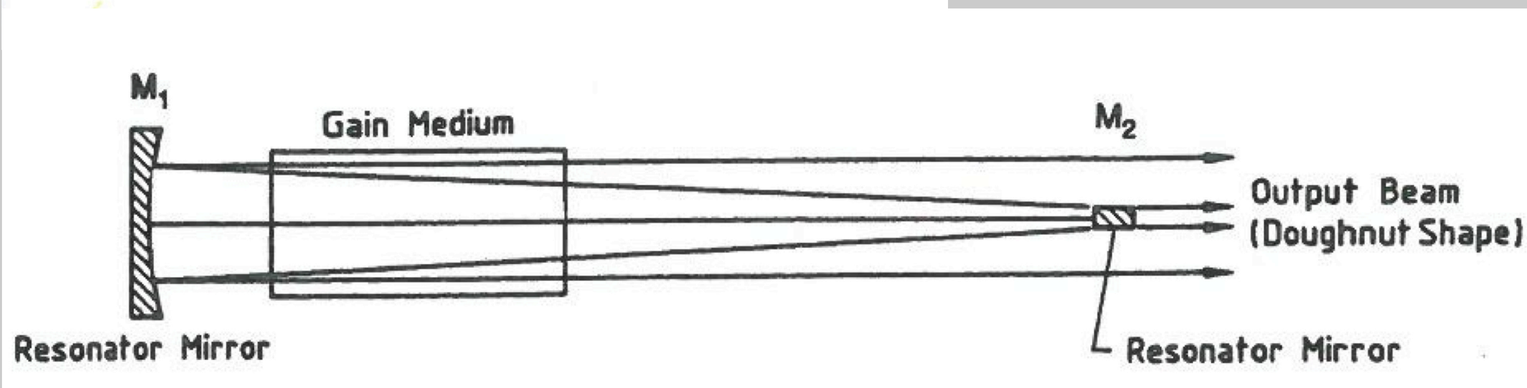


Telescope magnification:

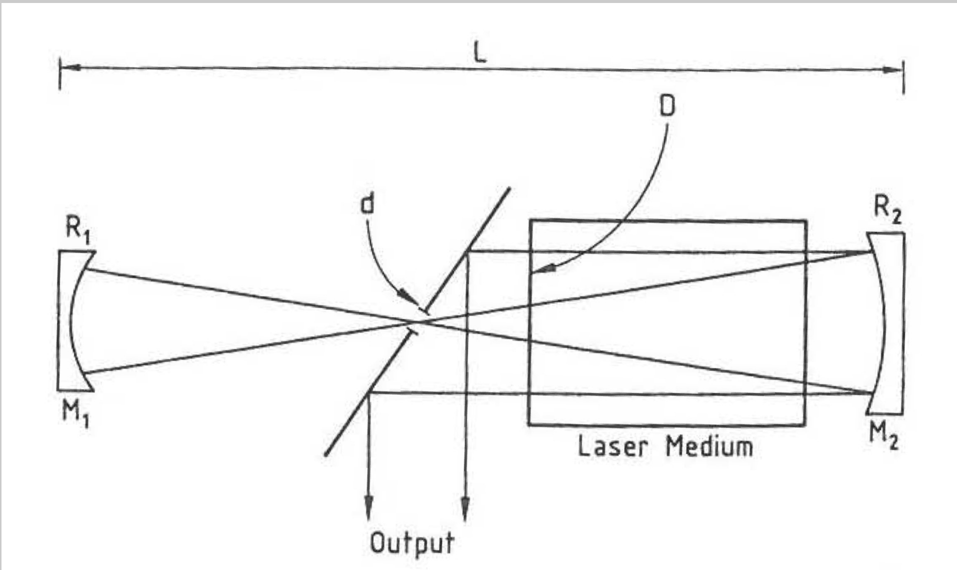
$$M = \frac{f_2}{f_1} = \frac{R_2}{R_1}$$

Output coupling loss per round trip:

$$\frac{A_2}{A_1} = M^2 = \left(\frac{R_2}{R_1}\right)^2$$

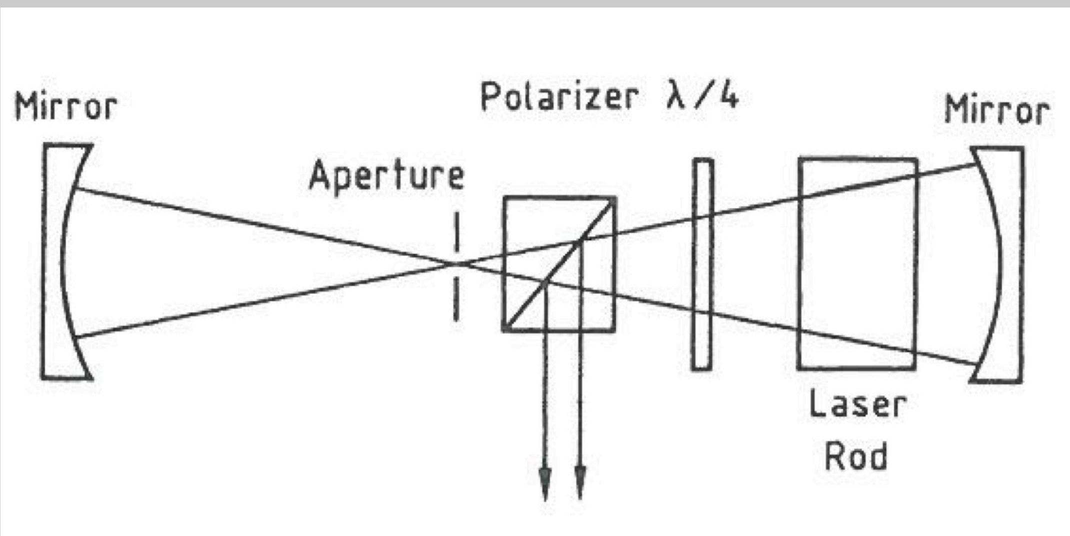


Negative branch unstable resonators



Aperture at intermediate focus
- Acts as an internal spatial filter

“scraper mirror” output



Polarization-coupled output

Zig-zag slab resonator

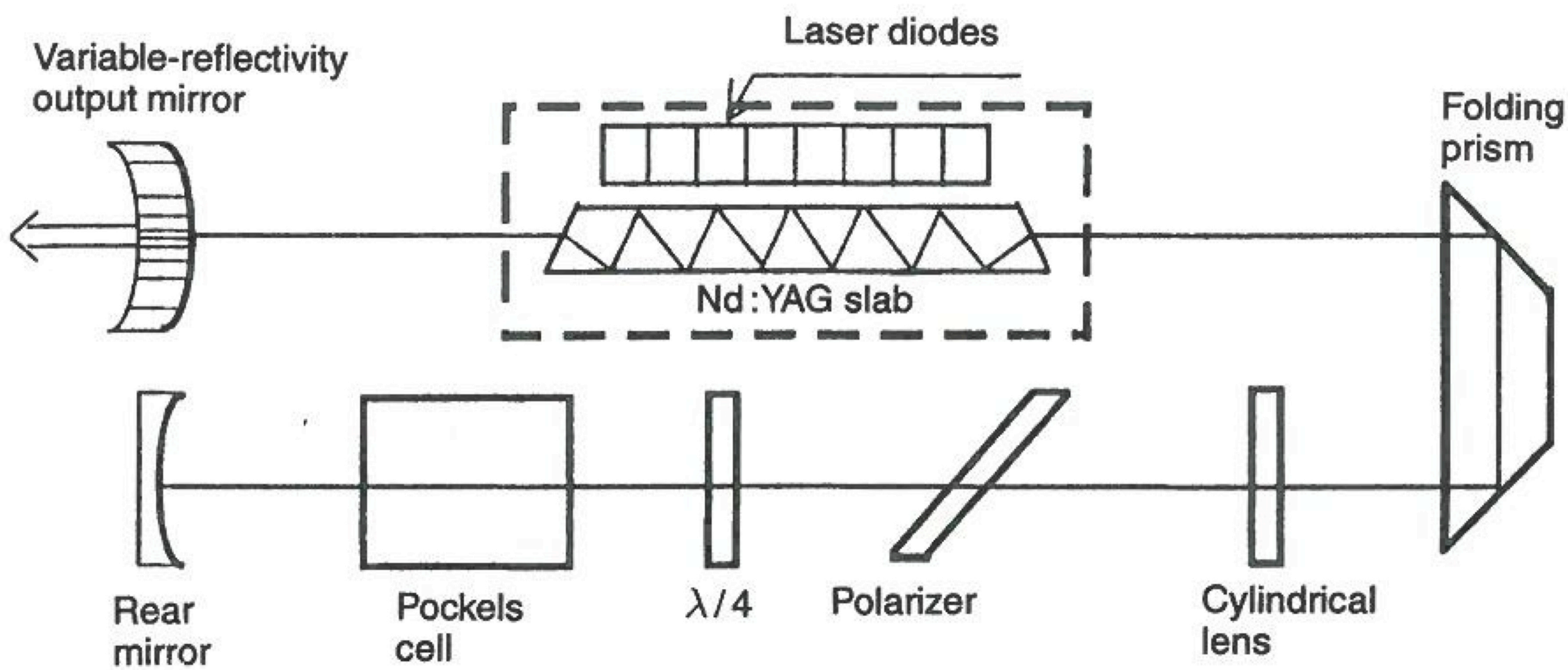


Fig. 5.50. Diode-pumped Nd:YAG slab laser with positive-branch unstable resonator and variable reflectivity output coupler [5.76]

Generalized ABCD

- Examples:
  - Variable output coupling mirrors
  - Radially-dependent gain
  - Parabolic refractive index profiles
  - Parabolic gain profiles – gain guiding
- ABCD with gain and loss lead to complex terms
  - Qualitative change to stability
  - Need additional modeling to calculate net gain and loss (ABCD is for beam shape, not amplitude)

Variable reflectivity mirror

- Gaussian mirror: graded reflectivity dielectric coating
- Beam curvature unaffected
- Beam size is reduced:

$$e^{-r^2/w_2^2} = e^{-r^2/w_1^2} e^{-r^2/w_m^2}$$