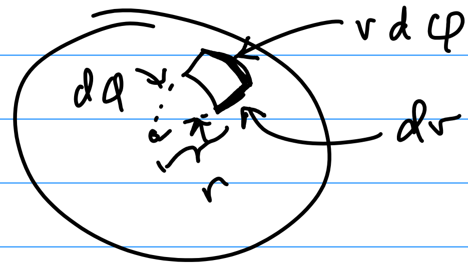
Solutions for homework assign #1

1.) area of a circle



$$da = r\,d\varphi\,dr$$

$$A = \int da = \int_0^R \int_0^{2\pi} r\,d\varphi\,dr = \pi R^2$$

2.) area of a sphere

$R\,d\theta$

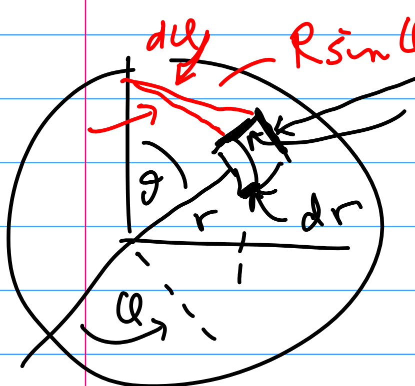
$R\sin\varphi\,d\varphi$

$$d_a = R\,d\theta\;R\sin\varphi\,d\varphi$$

$$A = \int da = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\varphi\,d\theta\,d\varphi = 4\pi R^2$$

3.) volume of a sphere

note r is inside sphere



$r\,d\theta$

$r\sin\varphi\,d\varphi$

$$V = \int dV = \int_0^R\int_0^{2\pi}\int_0^{\pi} R\,d\theta\,R\sin\varphi\,d\varphi\,dr = \frac{4}{3}\pi R^3$$

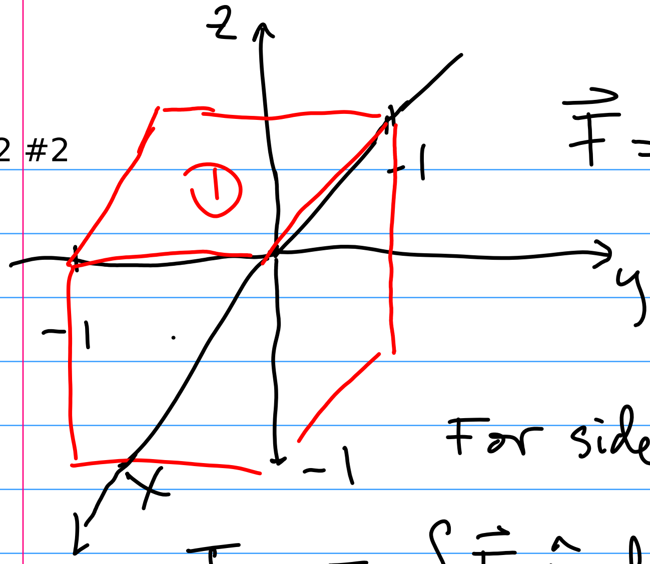
1-1 #3 $$\vec{\nabla}\varphi = \hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z} = y\hat{x} + x\hat{y}$$

1-1 #8 Use dot product to find angle

$$(\hat{x}-\hat{y})\cdot(2\hat{x}+\hat{y}) = \overbrace{\hat{x}\cdot 2\hat{x}}^{=1} - \hat{y}\cdot y = |\hat{x}-\hat{y}|\,|2\hat{x}+\hat{y}|\cos\varphi$$

$$= \sqrt{1^2+1^2}\,\sqrt{2^2+1^2}\cos\varphi \quad \text{solve for } \varphi$$

1-2 #2



$$\vec{F} = \hat{x}\, xy + \hat{y}\, yz + \hat{z}\, zx$$

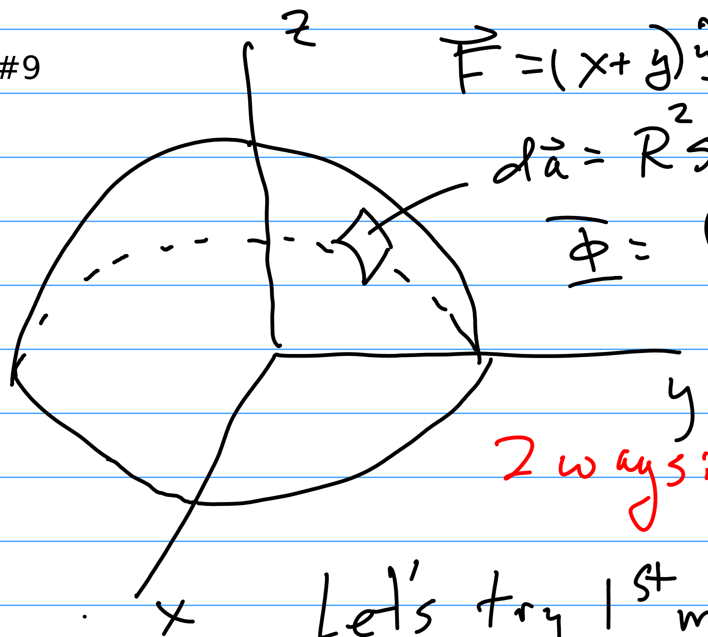
For side 1: $z = 0$ $\quad \hat{n} = \hat{z}$ $\quad \vec{F}\cdot\hat{n} = zx$

$$\Phi_{1} = \int \vec{F}\cdot\hat{n}\, dy\, dz\Big|_{z=0} = \int_{-1}^{0}\int_{-1}^{0} \underbrace{zx}_{\phi}\, dy\, dz = \phi$$

Repeat for all 6 sides

$$\Phi_{1} + \Phi_{2} + \Phi_{3} + \Phi_{4} + \Phi_{5} + \Phi_{6} = -3/2$$

1-2 #9



$$\vec{F} = (x+y)\hat{y} + (-x+y)\hat{y} + (-2z)\hat{z}$$

$$d\vec{a} = R^2 \sin\varphi\, d\varphi\, d\theta\, \hat{r}$$

$$\Phi = \int \vec{F}\cdot d\vec{a}$$

2 ways:

change $x, y, z \rightarrow r, \varphi, \theta$

change $r, \theta, \varphi \rightarrow x, y, z$

Let's try 1st method. Go to

http://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

Substitute these relations from the web page

$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$

$\hat{x} = \sin\theta\cos\varphi\,\hat{r} + \_\_\_\,\hat{\theta} + \_\_\_\,\hat{\varphi}$

$\hat{y} = \sin\theta\sin\varphi\,\hat{r} + \_\_\_\,\hat{\theta} + \_\_\_\,\hat{\varphi}$

$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

to change into $\vec{F} = (x+y)\hat{y} + (-x+y)\hat{y} + (-2z)\hat{z}$ & $d\vec{A}$

$$\Phi = \int \vec{F} \cdot d\vec{a} = \int \vec{F} \cdot \hat{r}\, R^2 \sin\theta\, d\theta\, d\varphi$$

Note that only the $\hat{r}$ component of F is non-zero

$$\vec{F} \cdot \hat{r} = (\underset{x}{r\sin\theta\cos\varphi} \underset{+}{+} \underset{y}{r\sin\theta\sin\varphi})\sin\theta\cos\varphi$$

$$+ (-r\sin\theta\cos\varphi + r\sin\theta\sin\varphi)\sin\theta\sin\varphi$$

$$+ (-2r\cos\theta)\cos\theta$$

$$\Phi = \int_0^{2\pi}\int_0^{\pi/2} \vec{F} \cdot \vec{r}\, R^2 \sin\theta\, d\theta\, d\varphi$$

1-3 #2 Easy to use $\vec{\nabla} \cdot$ in spherical coords

from http://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

for a vector function $\vec{V} = \underset{= \frac{1}{r^3}}{V_r}\hat{r} + \underset{=0}{V_\theta}\hat{\theta} + \underset{=0}{V_\varphi}\hat{\varphi}$

in spherical coords the

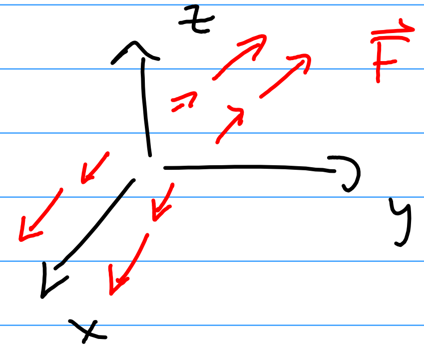
$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^3}\right)$$ NO θ or φ derivatives

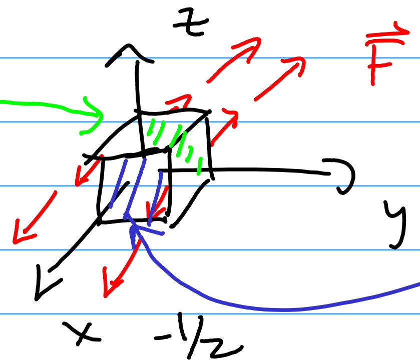
$$= \frac{1}{r^2}(-1)\frac{1}{r^2} = -\frac{1}{r^4}$$

1-4 #5

$$\underbrace{\oint \vec{F}\cdot d\vec{a}}_{LHS} = \overbrace{\int_{volume} \vec{\nabla}\cdot\vec{F}\,dx\,dy\,dz}^{RHS}$$

(a) $\vec{F} = x\hat{x}$





Flux only thru 2 sides

$$\Phi_1 = \iint \underset{\frac{1}{2}}{\underset{\parallel}{x}}\,\hat{x}\cdot dx\,dy\,\hat{x} = \frac{1}{2}\,\underset{1}{\underset{\parallel}{\text{area}}}$$

$$\Phi_2 = \iint \overset{-1/2}{\overset{\parallel}{x}}\,\hat{x}\cdot dx\,dy\,(-\hat{x}) = -\frac{1}{2}\,\overset{1}{\overset{\parallel}{\text{area}}}$$

$$\boxed{LHS = \Phi_1 + \Phi_2 = 1}$$

$$\iiint \vec{\nabla}\cdot\vec{F}\,dx\,dy\,dz = \iiint \overset{1}{\overset{\parallel}{\frac{\partial}{\partial x}x}}\,dx\,dy\,dz = \iiint_{-1/2}^{1/2} dx\,dy\,dz = 1$$

$$\boxed{RHS = 1}$$

problem 5 see lecture notes for

$$da' = R'^2 \sin\theta'\,d\varphi'\,d\theta' \qquad \vec{r} = z\hat{z}$$

$$\vec{r}\,' = R\hat{r} = R\sin\theta'\cos\varphi'\,\hat{x} + R\sin\theta'\sin\varphi'\,\hat{y} + R\cos\theta'\,\hat{z}$$

$$\vec{E} = \int \frac{k\,dq}{|\vec{r}-\vec{r}\,'|^2}\,\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$