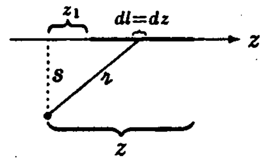
Problem 5.22

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{I\,\hat{\mathbf{z}}}{\mathcal{r}}\,dz = \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\int_{z_1}^{z_2}\frac{dz}{\sqrt{z^2+s^2}}$$

$$= \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]\Big|_{z_1}^{z_2} = \boxed{\frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{(z_2)^2+s^2}}{z_1+\sqrt{(z_1)^2+s^2}}\right]\hat{\mathbf{z}}}$$



$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A}{\partial s}\,\hat{\boldsymbol{\phi}} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{(z_2)^2+s^2}}\frac{s}{\sqrt{(z_2)^2+s^2}} - \frac{1}{z_1+\sqrt{(z_1)^2+s^2}}\frac{s}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2-\sqrt{(z_2)^2+s^2}}{(z_2)^2-[(z_2)^2+s^2]}\frac{1}{\sqrt{(z_2)^2+s^2}} - \frac{z_1-\sqrt{(z_1)^2+s^2}}{z_1^2-[(z_1)^2+s^2]}\frac{1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 I s}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - 1 - \frac{z_1}{\sqrt{(z_1)^2+s^2}} + 1\right]\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - \frac{z_1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\boldsymbol{\phi}},$$

or, since $\sin\theta_1 = \dfrac{z_1}{\sqrt{(z_1)^2+s^2}}$ and $\sin\theta_2 = \dfrac{z_2}{\sqrt{(z_2)^2+s^2}}$,

$$= \boxed{\frac{\mu_0 I}{4\pi s}(\sin\theta_2-\sin\theta_1)\,\hat{\boldsymbol{\phi}}}\ \text{(as in Eq. 5.35).}$$

Problem 7.2

(a) $V = Q/C = IR$. Because positive I means the charge on the capacitor is *decreasing*, $\dfrac{dQ}{dt} = -I = -\dfrac{1}{RC}Q$, so $Q(t) = Q_0 e^{-t/RC}$. But $Q_0 = Q(0) = CV_0$, so $\boxed{Q(t) = CV_0 e^{-t/RC}.}$

Hence $I(t) = -\dfrac{dQ}{dt} = CV_0\dfrac{1}{RC}e^{-t/RC} = \boxed{\dfrac{V_0}{R}e^{-t/RC}.}$

(b) $W = \boxed{\frac{1}{2}cV_0^2.}$ The energy delivered to the resistor is $\displaystyle\int_0^\infty P\,dt = \int_0^\infty I^2R\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2t/RC}dt = \frac{V_0^2}{R}\left(-\frac{RC}{2}e^{-2t/RC}\right)\Big|_0^\infty = \frac{1}{2}CV_0^2.$ ✓

(c) $V_0 = Q/C + IR$. This time positive I means Q is *increasing*: $\dfrac{dQ}{dt} = I = \dfrac{1}{RC}(CV_0 - Q) \Rightarrow \dfrac{dQ}{Q-CV_0} = -\dfrac{1}{RC}dt \Rightarrow \ln(Q-CV_0) = -\dfrac{1}{RC}t + \text{constant} \Rightarrow Q(t) = CV_0 + ke^{-t/RC}$. But $Q(0) = 0 \Rightarrow k = -CV_0$, so $\boxed{Q(t) = CV_0\left(1-e^{-t/RC}\right).}$ $I(t) = \dfrac{dQ}{dt} = CV_0\left(\dfrac{1}{RC}e^{-t/RC}\right) = \boxed{\dfrac{V_0}{R}e^{-t/RC}.}$

(d) Energy from battery: $\displaystyle\int_0^\infty V_0 I\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-t/RC}dt = \frac{V_0^2}{R}\left(-RCe^{-t/RC}\right)\Big|_0^\infty = \frac{V_0^2}{R}RC = \boxed{CV_0^2.}$

Since $I(t)$ is the same as in (a), the energy delivered to the resistor is again $\boxed{\frac{1}{2}CV_0^2.}$ The final energy in the capacitor is also $\boxed{\frac{1}{2}CV_0^2,}$ so $\boxed{\text{half}}$ the energy from the battery goes to the capacitor, and the other half to the resistor.