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MATH348 - April 18, 2012
Exam II - 50 Points

NAME: Key #1

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Modeling Concepts. For the following questions assume that we are considering the physical problem on a bounded domain, $x \in [0,1]$.

   (a) Write down the heat equation and any initial and boundary conditions needed to find a unique solution.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad \cancel{u(0,t) = g(t)}$$

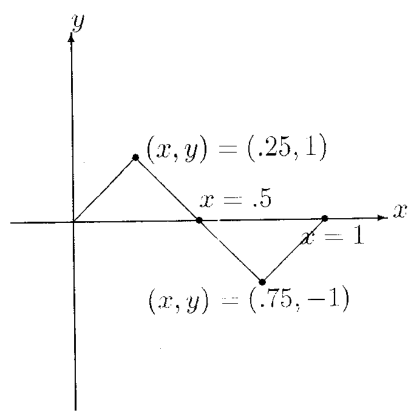
$$\alpha u(0,t) + \beta u_x(0,t) = g(t)$$

$$\gamma u(1,t) + \delta u_x(1,t) = h(t)$$

$$c^2 = \frac{K}{\rho\sigma}$$

   (b) Assume that the following graph is the initial temperature for a homogeneous heat problem with boundary conditions, $u_x(0,t) = 0$, $u_x(1,t) = 0$. Describe the physical meaning of these boundary conditions and graph the temperature profile for $t \to \infty$.

B.C. Imply total insulation at End pts.

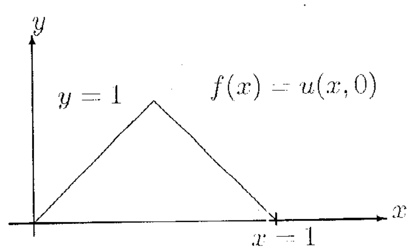


Since, for a closed System, as $t \to \infty$ the temp $\to$ avg temp over domain, in this Case temp $\to 0$ for all $x \in (0,1)$

   (c) The following graph gives the only nonzero initial configuration for the heat and wave equation. Describe and/or draw the associated heat **and** wave dynamics. If there is an equilibrium state then be sure to state it.

Heat:
In this case the insulation and stated dynamics gives temp $= 1/2$ as $t \to \infty$.

Along the way the triangle will ~~shall~~ flatten out to get there



Wave:
There is no Equilibrium State, without friction. The triangle will Oscillate about $y = 1/2$ with End pts being free.

2. (10 Points) Quick Questions

(a) The following table contains different boundary conditions for the ODE, $F'' + \lambda F = 0$, $\lambda \in [0, \infty)$. Fill in each table element with either a ✓ yes or a ✗ no.

	Boundary value problem has a cosine solution	Boundary value problem has a sine solution	Boundary value problem has a nontrivial constant solution
$F'(0) = 0,\ F(L) = 0$	✓	✗	✗
$F(0) = 0,\ F(L) = 0$	✗	✓	✗
$F'(0) = 0,\ F'(L) = 0$	✓	✗	✓
$F(0) = 0,\ F'(L) = 0$	✗	✓	✗

(b) Show that $u(x,t) = e^{it}\cos(x)$ satisfies the differential equation $iu_t = u_{xx}$.

$$\frac{\partial u}{\partial t} = i\,u(x,t), \quad \frac{\partial^2 u}{\partial x^2} = -u(x,t) \Rightarrow i\,u_t = u_{xx}$$

(c) Show that $u(x,t) = \dfrac{1}{x-t}$ satisfies the differential equation $u_t + u_x = 0$.

$$\frac{\partial u}{\partial t} = \frac{+1}{(x-t)^2}, \quad \frac{\partial u}{\partial x} = \frac{-1}{(x-t)^2} \Rightarrow u_t + u_x = 0$$

(d) Find the time ODE consistent with the PDE $u_t = u_{xx} + F(x,t)$ such that $u(0,t) = u(L,t) = 0$.

$$\Rightarrow F(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\left(\frac{n\pi}{L}x\right) \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) G_n$$

$$f_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$\Rightarrow u_t - u_{xx} - F(x,t) =$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\underbrace{\left[G_n' + \left(\frac{n\pi}{L}\right)^2 G_n - f_n(t)\right]} = 0$$

$$\Rightarrow G_n' + \left(\frac{n\pi}{L}\right)^2 G_n = f_n, \quad n = 1, 2, 3, \dots$$

3. (10 Points) Suppose you are given the following spatial and temporal solutions to a 1D-PDE,

$$G_n(t) = A_n e^{-t}\cos(\sqrt{\lambda_n - 1}t) + B_n e^{-t}\sin(\sqrt{\lambda_n - 1}t), \tag{1}$$

$$F_n(x) = \sin(\sqrt{\lambda_n}x), \quad \lambda_n = 2n\pi,\ n = 1, 2, 3, \ldots. \tag{2}$$

(a) Find the general solution to the PDE.

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n}x)\left[A_n e^{-t}\cos(\sqrt{\lambda_n - 1}\,t) + B_n e^{-t}\sin(\sqrt{\lambda_n - 1}\,t)\right]$$

(b) Solve for any unknown constants given $u(x, 0) = 0$ and $u_t(x, 0) = g(x)$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}x) = 0 \Rightarrow A_n = 0$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n}x)\left[-A_n e^{-t}\cos(\sqrt{\lambda_n - 1}\,t) - A_n\sqrt{\lambda_n - 1}\, e^{-t}\sin(\sqrt{\lambda_n - 1}\,t) - B_n e^{-t}\sin(t) + \sqrt{\ }B_n e^{-t}\cos(t)\right]$$

$$\Rightarrow u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin(\sqrt{\lambda_n}x) \Rightarrow B_n = \frac{2}{\sqrt{\lambda_n - 1}}\int_0^L g(x)\sin\left(\frac{\sqrt{\lambda_n - 1}}{\sqrt{\lambda_n}}x\right)$$

(c) Evaluate $\lim_{t\to\infty} u(x, t)$.

$u \to 0$ as $t \to \infty$

4. (10 Points) Find the three ODE consistent with $u_{tt} + .5u_t = u_{xx} + u_{yy}$.

$$u(x,t) = F(x,y)G(t) \Rightarrow u_{tt} + \frac{1}{2}u_t - u_{xx} - u_{yy} =$$

$$= G''F + \frac{1}{2}G'F - F_{xx}G - F_{yy}G = 0$$

$$\Leftrightarrow \frac{G'' + \frac{1}{2}G'}{G} = \frac{F_{xx} + F_{yy}}{F} = -\lambda \Rightarrow G'' + \frac{1}{2}G' + \lambda G = 0$$

$$F_{xx} + F_{yy} + \lambda F = 0$$

$$F(x,y) = X(x)Y(y)$$

$$\Rightarrow X''Y + XY'' + \lambda XY = 0$$

$$\Leftrightarrow \frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -k \Rightarrow X'' + kX = 0$$

$$Y'' + (\lambda - k)Y = 0$$

5. (10 Points) Find the unique solution to,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \begin{array}{l} x \in (0,\pi) \\ t \in (0,\infty) \end{array}, \tag{3}$$

$$u(0,t) = 0, \; u(\pi,t) = 0, \tag{4}$$

$$u(x,0) = 0. \tag{5}$$

Step 1: $u(x,t) = F(x)G(t) \Rightarrow (3) \Leftrightarrow \frac{G'}{G} = \frac{F''}{F} = -\lambda$

Step 2: $F''(x) + \lambda F(x) = 0$, $(4) \Rightarrow F(0) = 0,\; F(\pi) = 0$

$\Rightarrow F_n(x) = \sin(\sqrt{\lambda_n}\, x)$, $\sqrt{\lambda_n} = n$, $n = 1,2,3,\ldots$

thus $G_n(t) = A_n e^{-\lambda_n t}$, $A_n \in \mathbb{R}$

Step 3: $u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x)\, e^{-\lambda_n t}$

$u(x,0) = 0 = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x) \Rightarrow A_n = \frac{2}{L}\int_0^L \cancel{f(x)}^{\,0} \sin(\sqrt{\lambda_n}\, x)\,dx$

$= 0$

$\Rightarrow u(x,t) = 0$, which makes sense since the system is open and in thermal Equilibrium large

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NAME: Key #2

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1. (10 Points) Modeling Concepts. For the following questions assume that we are considering the physical problem on a bounded domain, $x \in [0, 1]$.

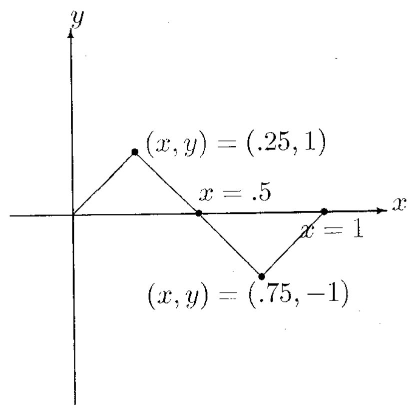
   (a) Write down the heat equation and any initial and boundary conditions needed to find a unique solution.

   $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad \alpha u(0,t) + \beta u_x(0,t) = g(t)$$

   $$c^2 = \frac{K}{\rho\sigma} \qquad \gamma u(1,t) + \delta u_x(1,t) = h(t)$$

   (b) Assume that the following graph is the initial temperature for a homogeneous heat problem with boundary conditions, $u(0,t) = 0$, $u(1,t) = 0$. Describe the physical meaning of these boundary conditions and graph the temperature profile for $t \to \infty$.

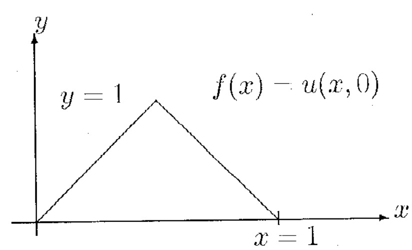
   B.C. imply the System is open + universe is 0°

   as $t \to \infty$ system will go to zero temp.

   (c) The following graph gives the only nonzero initial configuration for the heat and wave equation. Describe and/or draw the associated heat and wave dynamics. If there is an equilibrium state then be sure to state it.

   Heat:
   as $t \to \infty$
   Temp $\to 0$
   by Exp. decay of all Fourier mods.

   Wave.
   Triangle will Oscillate with the Ends fixed.
   There is no Equilibrium State

2. (10 Points) Quick Questions

(a) The following table contains different boundary conditions for the ODE, $F'' + \lambda F = 0$, $\lambda \in [0, \infty)$. Fill in each table element with either a yes or a no.

	Boundary value problem has a cosine solution	Boundary value problem has a sine solution	Boundary value problem has a nontrivial constant solution
$F(0) = 0,\ F(L) = 0$	No	Yes	no
$F'(0) = 0,\ F(L) = 0$	Yes	no	no
$F(0) = 0,\ F'(L) = 0$	no	Yes	no
$F'(0) = 0,\ F'(L) = 0$	~~no~~ Yes	~~Ye~~ no	Yes

(b) Show that $u(x,t) = f(x-t)$ satisfies the differential equation $u_{tt} = u_{xx}$.

$$\frac{\partial u}{\partial t} = \frac{\partial f(x-t)}{\partial t} = f'(x-t)\cdot(-1) \Rightarrow u_{tt} = f''$$

$$\frac{\partial u}{\partial x} = f'(x-t)\cdot 1 \Rightarrow u_{xx} = f'' \quad \Rightarrow u_{tt} = u_{xx}$$

(c) Show that $u(x,t) = e^{it}\cos(x)$ satisfies the differential equation $iu_t = u_{xx}$.

See Key #1

(d) Find the time ODE consistent with the PDE $u_t = u_{xx} + F(x,t)$ such that $u(0,t) = u(L,t) = 0$.

See Key #1

3. (10 Points) Given the following spatial solutions to a 1D-PDE,

$$F_n(x) = \cos(\sqrt{\lambda_n} x), \quad \sqrt{\lambda_n} = n, \; n = 0, 1, 2, \ldots, \tag{12}$$

where the temporal ODE is given by $G_n'' + \lambda_n G_n = 0$. $\Rightarrow$ $G_n(t) = A_n \cos(\sqrt{\lambda_n} t) + B_n \sin(\sqrt{\lambda_n} t)$ for $n \geq 1$

$G_0(t) = A_0 + B_0 t$

(a) Find the general solution to the PDE.

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x) \left[A_n \cos(\sqrt{\lambda_n} t) + B_n \sin(\sqrt{\lambda_n} t) \right]$$

(b) Solve for any unknown constants given $u(x, 0) = f(x)$.

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n} x) A_n \Rightarrow A_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos(\sqrt{\lambda_n} x)\, dx$$

B_n, B_0 are still unknown

(c) Evaluate $\lim_{t \to \infty} u(x, t)$.

There is no well defined limit state.

4. (10 Points) Find the three ODE consistent with $u_{tt} + .5u_t = u_{xx} + u_{yy}$.

See Key #1

5. (10 Points) Find the unique solution to,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \begin{matrix} x \in (0,\pi) \\ t \in (0,\infty) \end{matrix}, \tag{13}$$

$$u_x(0,t) = 0,\ u_x(\pi,t) = 0, \tag{14}$$

$$u(x,0) = f(x). \tag{15}$$

See Key #1 with the following changes.

(14) $\Rightarrow$ $F'(0) = F'(\pi) = 0 \Rightarrow F_n(x) = \cos(\sqrt{\lambda_n}x)$, $\sqrt{\lambda_n} = \frac{n\pi}{\pi}$, $n = \underline{\underline{0}}, 1, 2, \cdots$

$\Rightarrow G_n(t) = A_n e^{-\lambda_n t}$, $n = 0, 1, 2, \cdots$

thus,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n}x) e^{-\lambda_n t}$$

and

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n}x) e^{-\lambda_n t}$$

for A_0, A_n see prob 3 from this Exam.

space

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NAME: Key #3

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1. (10 Points) Modeling Concepts. For the following questions assume that we are considering the physical problem on a bounded domain, $x \in [0, 1]$.

   (a) Write down the wave equation and any initial and boundary conditions needed to find a unique solution.

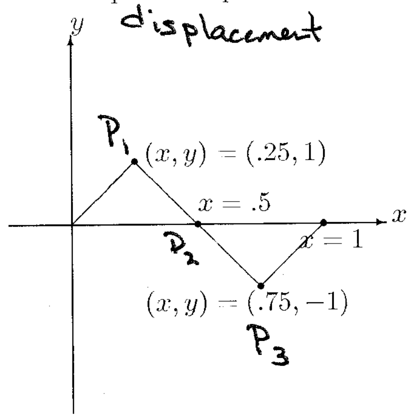
   $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho}$$

   $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, See Key #1, #2

   (b) Assume that the following graph is the initial displacement for a homogeneous wave problem with no initial velocity and boundary conditions, $u(0,t) = 0$, $u(1,t) = 0$. Describe the physical meaning of these boundary conditions and graph the ~~temperature~~ displacement profile for $t \to \infty$.

   As $t \to \infty$ the system continues to oscillate.

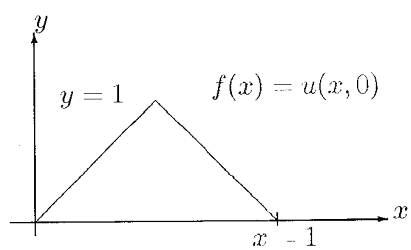
   P_1, P_3 oscillate while P_2 is a fixed node



   B.C. imply the End pts are fixed

   (c) The following graph gives the only nonzero initial configuration for the heat **and** wave equation. Describe and/or draw the associated heat **and** wave dynamics. If there is an equilibrium state then be sure to state it.

   See Key #2



2. (10 Points) Quick Questions

(a) The following table contains different boundary conditions for the ODE, $F'' + \lambda F = 0$, $\lambda \in [0, \infty)$. Fill in each table element with either a yes or a no.

	Boundary value problem has a cosine solution	Boundary value problem has a sine solution	Boundary value problem has a nontrivial constant solution
$F'(0) = 0,\ F(L) = 0$	✓	×	×
$F(0) = 0,\ F(L) = 0$	×	✓	×
$F(0) = 0,\ F'(L) = 0$	×	✓	×
$F'(0) = 0,\ F'(L) = 0$	✓	×	✓

(b) Show that $u(x,t) = \ln(x^2 + y^2)$ satisfies the differential equation $u_{xx} + u_{yy} = 0$.

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \quad u_{xx} + u_{yy} = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0$$

(c) Show that $u(x,t) = e^{it}\cos(x)$ satisfies the differential equation $iu_t = u_{xx}$.

See Key #1, #2

(d) Find the time ODE consistent with the PDE $u_t = u_{xx} + F(x,t)$ such that $u_x(0,t) = u_x(L,t) = 0$.

See Key #1

3. (10 Points) Given the following spatial and temporal solutions to a 1D-PDE,

$$G_n(t) = A_n e^{-\lambda_n t}, \tag{6}$$

$$F_n(x) = \cos(\sqrt{\lambda_n} x), \quad \lambda_n = n, \ n = 0, 1, 2, 3, \ldots. \tag{7}$$

(a) Find the general solution to the PDE.

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n t} \cos(\sqrt{\lambda_n} x)$$

(b) Solve for any unknown constants given $u(x,0) = f(x)$.

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n} x)$$

See Key #2

(c) Evaluate $\lim_{t\to\infty} u(x,t)$.

$$\lim_{t\to\infty} u = A_0 = f_{avg}$$

4. (10 Points) Find the three ODE consistent with $u_{tt} + .5u_t = u_{xx} + u_{yy}$.

See Key #1

5. (10 Points) Find the unique solution to,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad \begin{matrix} x \in (0,\pi) \\ t \in (0,\infty) \end{matrix}, \tag{8}$$

$$u(0,t) = 0,\ u(\pi,t) = 0, \tag{9}$$

$$u(x,0) = f(x), \tag{10}$$

$$u_t(x,0) = 0. \tag{11}$$

See Key#1 with the following change

$$G_n'' + \lambda_n G = 0 \Rightarrow G_n(t) = A_n \cos(\sqrt{\lambda_n}\, t) + B_n \sin(\sqrt{\lambda_n}\, t)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n}\, x)\left[A_n \cos(\sqrt{\lambda_n}\, t) + B_n \sin(\sqrt{\lambda_n}\, t)\right]$$

$$(10) \Rightarrow \quad f(x) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x) \Rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}\, x)\,dx$$

$$(11) \Rightarrow \quad 0 = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, B_n \sin(\sqrt{\lambda_n}\, x) \Rightarrow B_n = 0$$