10

Harmonic generation with focused beams
Fourier transforms
pulse propagation

Harmonic generation with focused Gaussian beams

- q[th] harmonic:
  $\omega_q = q\omega_1 \qquad I_q(r) \propto I_1^q(r)$
- Spot size is smaller: $w_q = w_1 / \sqrt{q}$
- Rayleigh range is the same:
  $$z_{Rq} = \frac{\pi w_q^{\,2}}{\lambda_q} = \frac{\pi \left(w_q / \sqrt{q}\right)^2}{\lambda_q / q} = z_{R1}$$
- no depletion
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- *Assume* harmonic propagates as a TEM_{00} beam

$$A_q(r,z) = A_{q0}(z) \frac{1}{1+i\xi_q} e^{-\frac{r^2}{w_{q0}^{\,2}\left(1+i\xi_q\right)}} \qquad \xi \equiv \frac{z}{z_R}$$

Develop equation for $A_q(z)$

- Gaussian beam is a solution for wave equation, so homogeneous part of equation (LHS) is

$$A_{q0}(z)\left(2ik_q\frac{\partial}{\partial z}+\nabla_\perp^{\ 2}\right)\frac{1}{1+i\xi_q}e^{-\frac{r^2}{w_{q0}^{\ 2}(1+i\xi_q)}}+\frac{1}{1+i\xi_q}e^{-\frac{r^2}{w_{q0}^{\ 2}(1+i\xi_q)}}\frac{\partial}{\partial z}A_{q0}(z)$$

- Simplify wave equation:

$$\frac{1}{1+i\xi_q}e^{-\frac{r^2}{w_{q0}^{\ 2}(1+i\xi_q)}}\frac{\partial}{\partial z}A_{q0}(z)=-\frac{\omega_q^{\ 2}}{2ik_q\varepsilon_0c^2}\chi^{(q)}\left(\frac{1}{1+i\xi_1}\right)^q e^{-\frac{qr^2}{w_0^{\ 2}(1+i\xi_1)}}A_1^{\ q}e^{i\Delta kz}$$

$$\frac{\partial}{\partial z}A_{q0}(z)=-\frac{\omega_q^{\ 2}}{2ik_q\varepsilon_0c^2}\chi^{(q)}\frac{1+i\xi_q}{(1+i\xi_1)^q}e^{-\frac{qr^2}{w_0^{\ 2}(1+i\xi_1)}+\frac{r^2}{w_{q0}^{\ 2}(1+i\xi_q)}}A_1^{\ q}e^{i\Delta kz}$$

Phase matching integral, non-depleted limit

- Both fundamental and harmonics are Gaussian beams with matched Rayleigh ranges

$$\frac{\partial}{\partial z}A_{q0}(z)=-\frac{\omega_q^{\ 2}}{2ik_q\varepsilon_0c^2}\chi^{(q)}\frac{1+i\xi_q}{(1+i\xi_1)^q}e^{-\frac{qr^2}{w_0^{\ 2}(1+i\xi_1)}+\frac{r^2}{w_{q0}^{\ 2}(1+i\xi_q)}}A_1^{\ q}e^{i\Delta kz}$$

  - Assume $n_q = n_1$ (e.g. gas) Since

$$z_R(\omega_q)=z_R(\omega_1)\qquad w_{q0}^2=w_0^2/q\qquad \xi\equiv\frac{z}{z_R}\qquad \xi_q=\xi_1$$

  - Equation for Aq(z) simplifies

$$\rightarrow\frac{\partial}{\partial z}A_{q0}(z)=i\frac{\omega_q}{2n_q\varepsilon_0c}\chi^{(q)}\frac{1}{(1+i\xi)^{q-1}}A_1^{\ q}e^{i\Delta kz}$$

New phase matching integral

- We start with equation for A_q:

$$\frac{\partial}{\partial z}A_{q0}(z) = i\frac{\omega_q}{2n_q\varepsilon_0 c}\chi^{(q)}\frac{1}{(1+iz/z_R)^{q-1}}A_1^{\,q}e^{i\Delta k z}$$

  – Integrate this directly

$$\rightarrow A_{q0}(z) = i\frac{\omega_q}{2n_q\varepsilon_0 c}\chi^{(q)}A_1^{\,q}J_q(z)$$

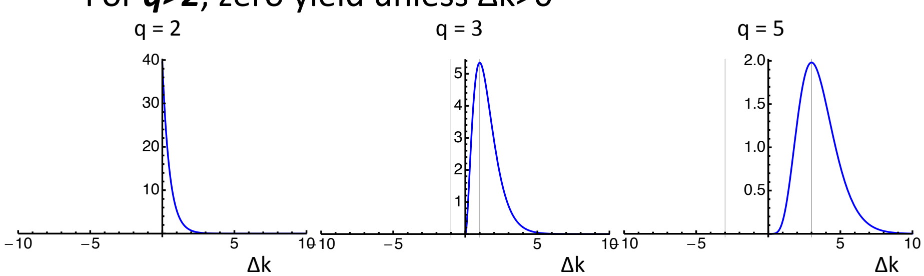
$$J_q(z) = \int_{z_1}^{z_2}\frac{1}{(1+iz'/z_R)^{q-1}}e^{i\Delta k z'}dz'$$

Harmonic generation in the tight focusing limit

- Here we integrate over all z:

$$J_q(\Delta k, z_R) = \begin{cases} 0 & \Delta k < 0 \\ z_R\dfrac{2\pi}{(q-2)!}(\Delta k\, z_R)^{q-2}e^{-\Delta k\, z_R} & \Delta k \geq 0 \end{cases}$$

- For ***q>2***, zero yield unless Δk>0



Small thickness limit

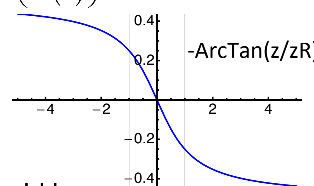
$$J_q(z) = \int_{-L/2}^{L/2} \frac{1}{(1 + i z'/z_R)^{q-1}} e^{i\Delta k z'} dz'$$

The fraction in the integrand is connected to the Gouy phase:

$$\frac{1}{1+i\xi} = \frac{1}{1+iz/z_R} = \frac{w_0}{w(z)} e^{-i\eta(z)} \qquad \eta(z) = \arctan\left(\frac{z}{z_R}\right)$$

$$\frac{1}{(1+iz'/z_R)^{q-1}} = \frac{1}{(1+i\xi)^{q-1}} = \left(\frac{w_0}{w(z)}\right)^{q-1} e^{-i(q-1)\eta(z)} \approx \left(\frac{w_0}{w(z)}\right)^{q-1} e^{-i(q-1)z/z_R}$$

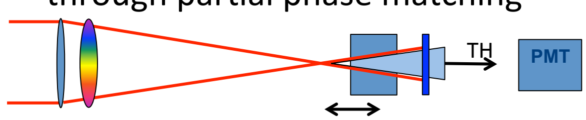
$$J_q(z) = \int_{-L/2}^{L/2} \left(\frac{w_0}{w(z)}\right)^{q-1} e^{i(\Delta k - (q-1)/z_R)z'} dz'$$



This shows where the optimum phase mismatch should be.
This limit is related to HG in waveguides, since the WG phase scales like z/zR

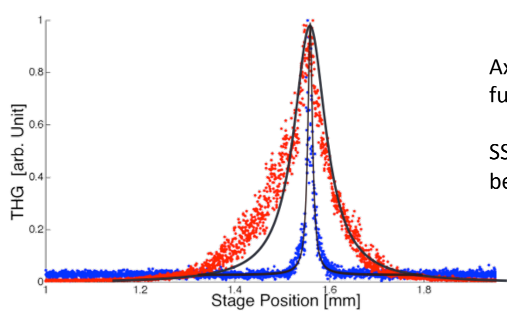
Measuring localization with THG

- Z-scan of fused silica interface leads to observed THG through partial phase matching



No THG from bulk

THG emerges with spatial chirp



Axial FWHM = confocal parameter of fundamental in air

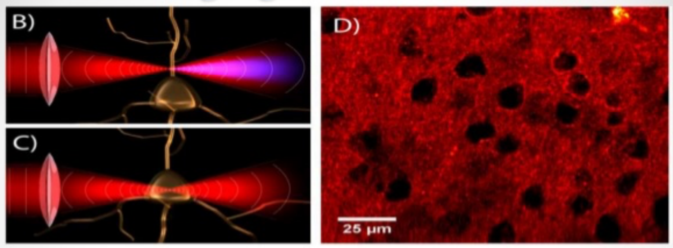
SSTF reduces FWHM consistent with beam aspect ratio

E. Block
O. Masihzadeh
C. Durfee
J. Squier

THG microscopy

- Label/dye-free signal. *Witte et al PNAS v108, 5970 2011*

THG Imaging of Live Brain Tissue



[B] **Axons** and **dentrites** have very **high lipid concentrations**, small diameters (0.3 – 2 μm)

- Laser focal volume >> diameter : **very good** THG signal

[C] Neuron cell bodies (somata) contain organelles, much smaller diameters (20 – 100 nm)

- THG signal **not** good

[D] Result: "**shadow contrast**" : neural tissue THG image shows **dark cell bodies** and **illuminated axons** / **dentrites**

Fourier transforms: t-ω domain

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{+i\omega t}\,dt = FT\left\{f(t)\right\}$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}\,dt = FT^{-1}\left\{F(\omega)\right\}$$

- In EM, our signals are complex fields
- $1/2\pi$ factor is lumped into inverse transform
- ω is our frequency variable, not ν. This affects the normalization constants
- Note signs of exponents: this is tied to our exp(-i ω t) convention
- Techniques
  - Analytic: apply transform IDs and theorems to decompose a transform into its parts
  - Analytic in Mathematica: can do some FTs but not always expressed in recognizable way
  - Graphical: after identifying components of a transform, sketch the anticipated result
  - Numerical: FFT for calculating complicated or realistic cases for modeling/data analysis

FT of a Gaussian

- Starting integral: $\int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\pi}$
  - True even if z is complex

$$f(t) = e^{-t^2/t_0^2} \qquad FT\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t}\,dt$$

- Complete the square in the exponent…

FT of a Gaussian is a Gaussian

- Starting integral: $\int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\pi}$
  - True even if z is complex

$$f(t) = e^{-t^2/t_0^2} \qquad FT\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t}\,dt$$

- Complete the square in the exponent

$$-\frac{t^2}{t_0^2} + i\omega t = -\frac{1}{t_0^2}\left(t^2 - i\,\omega\,t\,t_0^2\right) = -\frac{1}{t_0^2}\left(\left(t - \frac{i}{2}\,\omega\,t_0^2\right)^2 + \frac{1}{4}\,\omega^2\,t_0^4\right)$$

$$= -\frac{1}{t_0^2}\left(t - \frac{i}{2}\,\omega\,t_0^2\right)^2 - \frac{1}{4}\,\omega^2\,t_0^2$$

  - Change variables: $z = \frac{1}{t_0}\left(t - \frac{i}{2}\,\omega\,t_0^2\right)$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t}\,dt = t_0 e^{-\frac{1}{4}\omega^2 t_0^2} \int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\pi}\,t_0\,e^{-\frac{1}{4}\omega^2 t_0^2}$$

Time-bandwidth product

- "uncertainty principle" comes from FT relations

  $$FT\left(e^{-t^2/t_0^2}\right) \to t_0\, e^{-\frac{1}{4}\omega^2 t_0^2}$$

  – Pulse duration: t_0
  – Spectral width (bandwidth): $\delta\omega = 2/t_0$
  – Time-bandwidth product: $t_0\delta\omega = 2$

- This relation depends on how widths are defined
  – Here we've been using 1/e half width in the field
  – For FWHM in intensity: $E(t) = E_0 e^{-2\ln 2\, t^2/\tau^2} \to I(t) \propto e^{-4\ln 2\, t^2/\tau^2}$

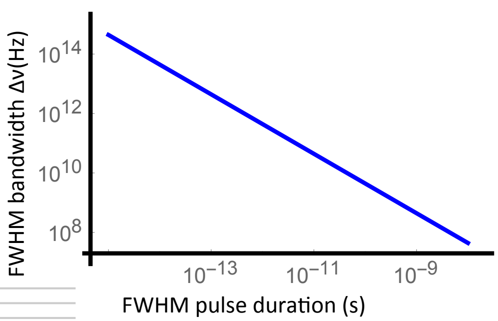
  $$\tau = t_0\sqrt{2\ln 2} \qquad \Delta\omega = \delta\omega\sqrt{2\ln 2}$$

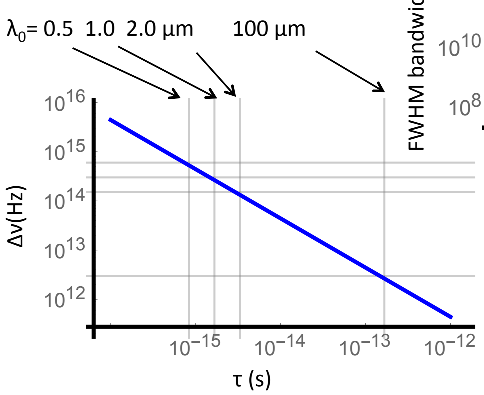
  $$t_0\delta\omega = 2 = \frac{\tau\Delta\omega}{2\ln 2} \to \tau\Delta\omega = 4\ln 2 \approx 2.77$$

  $$\tau\,\Delta\nu = \frac{4\ln 2}{2\pi} \approx 0.44$$

Bandwidth for transform-limited pulses

- The bandwidth in frequency space is independent of the central frequency





The shortest pulse is limited to ½ of an optical cycle

FT(rect)=sinc and Dirac delta

- Rect(t/t0) $rect\left(\frac{t}{t_0}\right)=1 \text{ for } |t|<\frac{t_0}{2}$

$$F(\omega)=\int_{-\infty}^{\infty} rect(t/t_0)e^{+i\omega t}\,dt=\int_{-t_0/2}^{t_0/2} e^{+i\omega t}\,dt=\frac{1}{i\omega}\left(e^{+i\omega t_0/2}-e^{-i\omega t_0/2}\right)$$

$$=t_0\frac{\sin(\omega t_0/2)}{\omega t_0/2}=t_0\,\text{sinc}(\omega t_0/2)$$

- Dirac delta $\int_{-\infty}^{\infty}\delta(t)dt=1$
  - Limit:

$$\delta(\omega)=\lim_{t_0\to\infty} FT\{\text{rect}(t/t_0)\}=\lim_{t_0\to\infty}\left[t_0\,\text{sinc}(\omega t_0/2)\right]$$

  - At ω=0, limit is ∞
  - ω≠0, limit is 0 in sense that integral over rapid osc sin( ) is 0
  - Normalization:

$$FT\{1\}=2\pi\delta(\omega) \qquad FT^{-1}\{1\}=\delta(t)$$

FT theorems

Properties of Fourier Transforms

A_1 and A_2 arbitrary constants; x_0 and ξ_0 real constants

b and d real nonzero constants; k a positive integer

$g(x)=\int_{-\infty}^{\infty}G(\beta)e^{j2\pi\beta x}d\beta$	$G(\xi)=\int_{-\infty}^{\infty}g(\alpha)e^{-j2\pi\alpha\xi}d\alpha$	
$f(\pm x)$	$F(\pm\xi)$	
$f^*(\pm x)$	$F^*(\mp\xi)$	
$F(\pm x)$	$f(\mp\xi)$	
$F^*(\pm x)$	$f^*(\pm\xi)$	
$f\left(\frac{x}{b}\right)$	$\lvert b\rvert F(b\xi)$	**scaling**
$\lvert d\rvert f(dx)$	$F\left(\frac{\xi}{d}\right)$	
$f(x\pm x_0)$	$e^{\pm j2\pi x_0\xi}F(\xi)$	**shift**
$e^{\pm j2\pi\xi_0 x}f(x)$	$F(\xi\mp\xi_0)$	

FT Theorems

Shift theorem

$$\Im\{f(t-t_0)\} = \exp(+i\omega t_0)F(\omega) \qquad \Im^{-1}\{F(\omega-\omega_0)\} = \exp(-i\omega_0 t)f(t)$$

Scale theorem

$$\Im\{f(at)\} = \frac{1}{|a|}F(\omega/a) \qquad \Im^{-1}\{F(b\omega)\} = \frac{1}{|b|}f(t/b)$$

Conjugation

$$\Im\{f^*(t)\} = F^*(-\omega)$$

Symmetry properties of FT

Symmetry Properties of Fourier Transforms

$f(x)$	$F(\xi)$
Complex, no symmetry	Complex, no symmetry
Hermitian	Real, no symmetry
Antihermitian	Imaginary, no symmetry
Complex, even	Complex, even
Complex, odd	Complex, odd
Real, no symmetry	Hermitian
Real, even	Real, even
Real, odd	Imaginary, odd
Imaginary, no symmetry	Antihermitian
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd

Representing an optical pulse in t and ω spaces

- Two ways to represent the field of a pulse:
  - Time domain

$$E(t)=A(\mathbf{r},t)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}-i\,\omega_0\,t+\phi(t)\right)\right]+c.c.$$

  - Frequency domain

$$E(\omega)=FT\left\{E(t)\right\}=A(r,\omega-\omega_0)\,e^{i\left(\mathbf{k}\cdot\mathbf{r}+\varphi(\omega-\omega_0)\right)}+A^*(r,\omega+\omega_0)\,e^{-i\left(\mathbf{k}\cdot\mathbf{r}-\varphi(\omega-\omega_0)\right)}$$

  - Both positive and negative frequency components: usually neglect negative side in linear optics

$$E(\omega)\approx A(r,\omega-\omega_0)\exp\left[i\left(\mathbf{k}\cdot\mathbf{r}+\varphi(\omega-\omega_0)\right)\right]$$

- Both t and ω representations contain the same information, same total energy.
- **Phase functions *not the same in both domains***
- **Temporal phase:** $\phi(t)$
- **Spectral phase:** $\varphi(\omega)$

Taylor expansion of spectral phase

- To simplify the phase, consider the first two terms

$$\varphi(\omega)\;=\;\varphi_0\;+\;\varphi_1\,\frac{\omega-\omega_0}{1!}\;+\;\varphi_2\,\frac{(\omega-\omega_0)^2}{2!}\;+\;\ldots$$

where $\varphi_0=\varphi(\omega_0)$ is the **"absolute phase"**

$\varphi_1=\left.\frac{d\varphi}{d\omega}\right|_{\omega=\omega_0}$ is the **group delay**. $\tau_g(\omega)=\frac{d\varphi}{d\omega}$

$\varphi_2=\left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega=\omega_0}$ is called the "**group-delay dispersion**."

- In real situations, we sometimes have to include higher order phase, 3^{rd}, 4^{th}…

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Taylor expansion of temporal phase

- Here we expand around t=0, i.e. the center of the pulse

$$\phi(t) = \phi_0 + \phi_1 \frac{t}{1!} + \phi_2 \frac{t^2}{2!} + \ldots$$

where $\phi_0 = \phi(0)$ is the **"carrier-envelope" or "absolute phase"**

$\phi_1 = \left.\frac{d\phi}{dt}\right|_{t=0}$ the **instantaneous frequency is** $\omega_{inst}(t) = -\frac{d\phi}{dt}$

$\phi_2 = \left.\frac{d^2\phi}{dt^2}\right|_{t=0}$ is called the "**temporal chirp**."

- In real situations, we sometimes have to include higher order phase, 3rd, 4th…

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Intensity and phase of a Gaussian

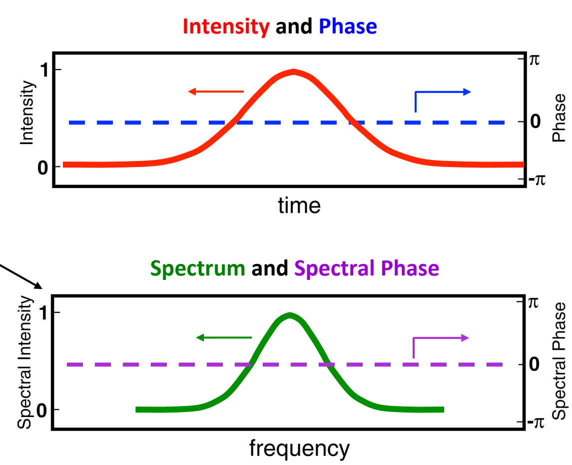
- The Gaussian is real, so its phase is zero in both domains.

Time domain:

A Gaussian transforms to a Gaussian

Frequency domain:

So the spectral phase is zero, too.



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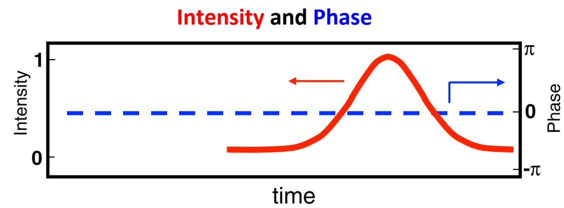
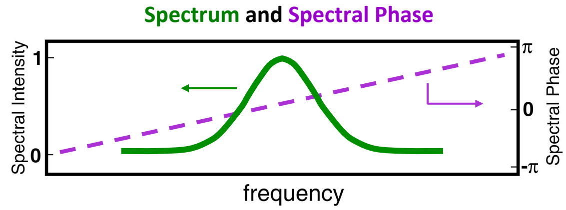
The spectral phase of a time-shifted pulse

Recall the Shift Theorem: $FT\{f(t-t_0)\} = \exp(+i\omega t_0)F(\omega)$

Time-shifted Gaussian pulse (with a flat phase):



So a time-shift simply adds some linear spectral phase to the pulse!



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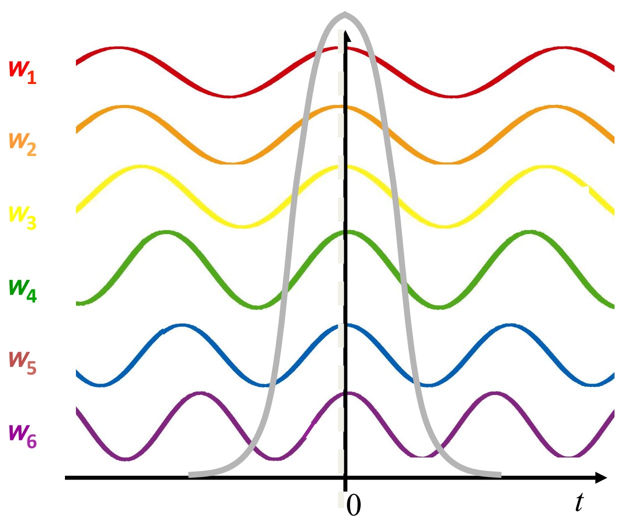
What is the spectral phase?

The spectral phase is the phase of each frequency in the wave-form.



All of these frequencies have zero phase. So this pulse has:

$\phi(\omega) = 0$

Note that this wave-form sees constructive interference, and hence peaks, at $t = 0$.

And it has cancellation everywhere else.

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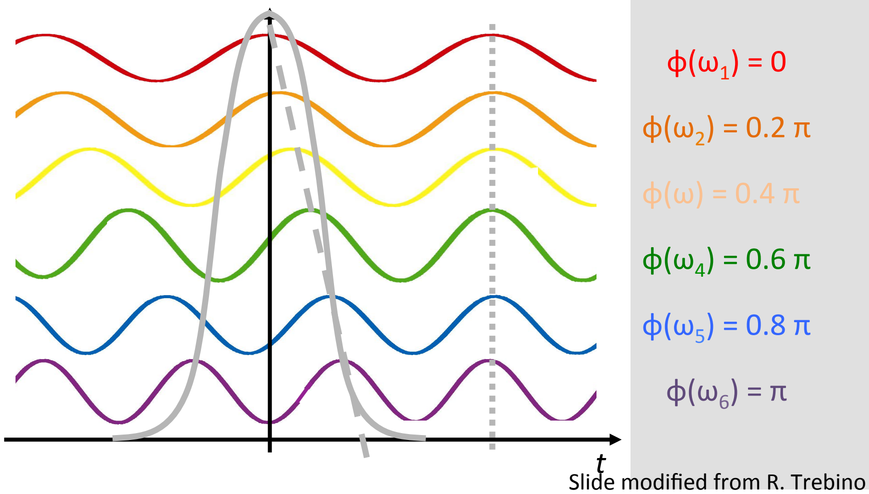
Linear spectral phase: $\phi(\omega) = a\omega$.

By the Shift Theorem, a linear spectral phase is just a delay in time.

The peaks of the spectral components line up at a later time.



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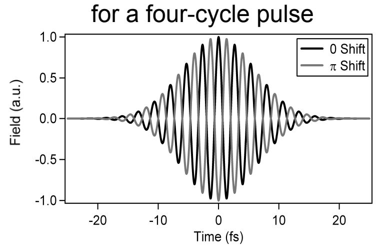
Zero[th]-order phase: the absolute phase

- The absolute phase is the same in both the time and frequency domains.

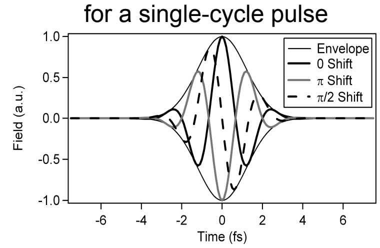
$$f(t)\exp(i\phi_0) \rightarrow F(\omega)\exp(i\phi_0)$$

- An absolute phase of $\pi/2$ will turn a cosine carrier wave into a sine.
- It's usually irrelevant, unless the pulse is only a cycle or so long.

Different absolute phases for a four-cycle pulse



Different absolute phases for a single-cycle pulse



Notice that the two four-cycle pulses look alike, but the three single-cycle pulses are all quite different.

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First-order phase in frequency: a shift in time

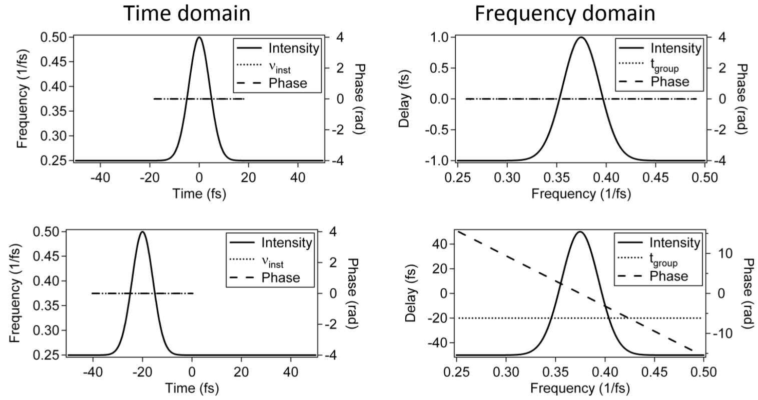
By the Fourier-Transform Shift Theorem,

$$F(\omega)\exp(i\omega\varphi_1) \rightarrow f(t-\varphi_1)$$

Time domain — Frequency domain

$\varphi_1 = 0$

$\varphi_1 = -20$fs



Note that φ_1 does not affect the instantaneous frequency, but the group delay = ϕ_1.

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First-order phase in time: a frequency shift

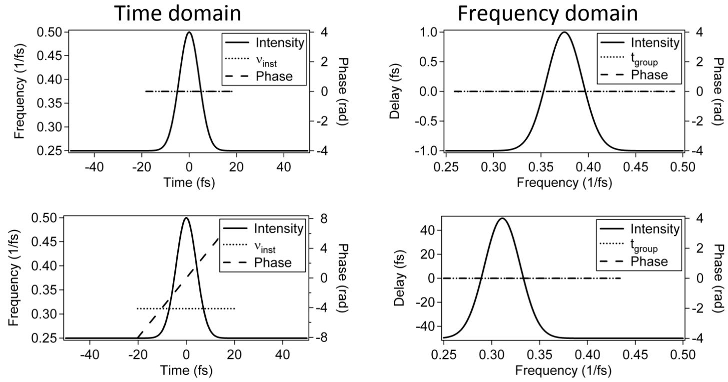
- By the Inverse-Fourier-Transform Shift Theorem,

$$f(t)\exp(-i\phi_1 t) \rightarrow F(\omega-\phi_1)$$

Time domain — Frequency domain

$\phi_1 = 0/$fs

$\phi_1 = -.07/$fs



Note that ϕ_1 does not affect the group delay, but it does affect the instantaneous frequency.

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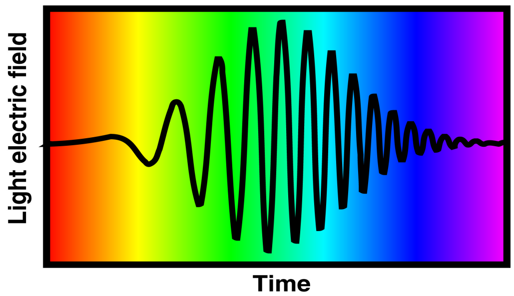
Second-order phase: the linearly chirped pulse

- A pulse can have a frequency that varies in time.



This pulse increases its frequency linearly in time (from red to blue).

In analogy to bird sounds, this pulse is called a "chirped" pulse.

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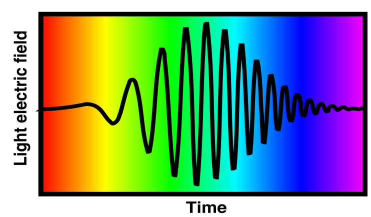
The linearly chirped Gaussian pulse



- We can write a linearly chirped Gaussian pulse mathematically as:

$$E(t) = A(t)\exp\left[i\,\phi(t)\right]$$
$$= E_0 \exp\left[-(t/\tau_G)^2\right]\exp\left[-i\left(\omega_0 t + \beta t^2\right)\right]$$

↑ Gaussian amplitude   Carrier wave ↑   ↑ Chirp

Note that for $\beta > 0$, when $t < 0$, the two terms partially cancel,
so the phase changes slowly with time (so the frequency is low).
And when $t > 0$, the terms add, and the phase changes more rapidly
(so the frequency is larger).

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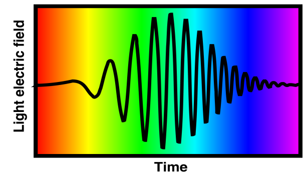
The instantaneous frequency vs. time for a chirped pulse



A chirped pulse has:

$$E(t) \propto \exp\left[i\left(-\omega_0 t + \phi(t)\right)\right]$$

where:

$$\phi(t) = -\beta t^2 \qquad \text{(note the sign change)}$$

The instantaneous frequency is:

$$\omega_{inst}(t) \equiv \omega_0 - d\phi / dt$$

which is:

$$\omega_{inst}(t) = \omega_0 + 2\beta t$$

So the frequency increases linearly with time. This is *positive* chirp.

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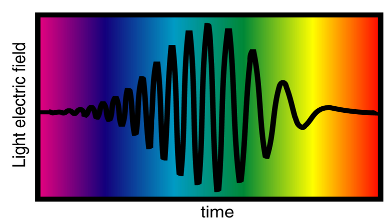
The negatively chirped pulse

- We have been considering a pulse whose frequency *increases*
- linearly with time: a *positively* chirped pulse.

- One can also have a *negatively*
- chirped (Gaussian) pulse, whos
- instantaneous frequency
- *decreases* with time.



- We simply allow β to be negati
- in the expression for the pulse:

$$E(t) = E_0 \exp\left[-\left(t/\tau_G\right)^2\right] \exp\left[-i\left(\omega_0 t + \beta t^2\right)\right]$$

$$= E_0 \exp\left[-\left(t/\tau_G\right)^2\right] \exp\left[-i\left(\omega_0 t - |\beta| t^2\right)\right]$$

- And the instantaneous frequency will decrease with time:

$$\omega_{inst}(t) = \omega_0 + 2\beta t = \omega_0 - 2|\beta| t$$

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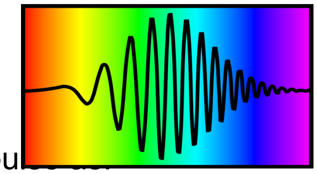
The Fourier transform of a chirped pulse



- Writing a linearly chirped Gaussian pulse as:

$$\mathscr{E}(t) \propto E_0 \exp\left[-\alpha t^2\right]\exp\left[-i\left(\omega_0 t + \beta t^2\right)\right] + c.c \quad \text{where} \quad \alpha \propto 1/\Delta t^2$$

- or:

$$\mathscr{E}(t) \propto E_0 \exp\left[-\left(\alpha + i\beta\right)t^2\right]\exp\left[-i\,\omega_0 t\right] + c.c.$$

- Fourier-Transforming yields:

neglecting the negative-frequency term due to the c.c.

$$\tilde{E}(\omega) \propto E_0 \exp\left[-\frac{1/4}{\alpha + i\beta}\left(\omega - \omega_0\right)^2\right]$$

- Rationalizing the denominator and separating the real and imag parts:

$$\tilde{E}(\omega) \propto E_0 \exp\left[-\frac{\alpha/4}{\alpha^2+\beta^2}\left(\omega-\omega_0\right)^2\right]\exp\left[+i\frac{\beta/4}{\alpha^2+\beta^2}\left(\omega-\omega_0\right)^2\right]$$

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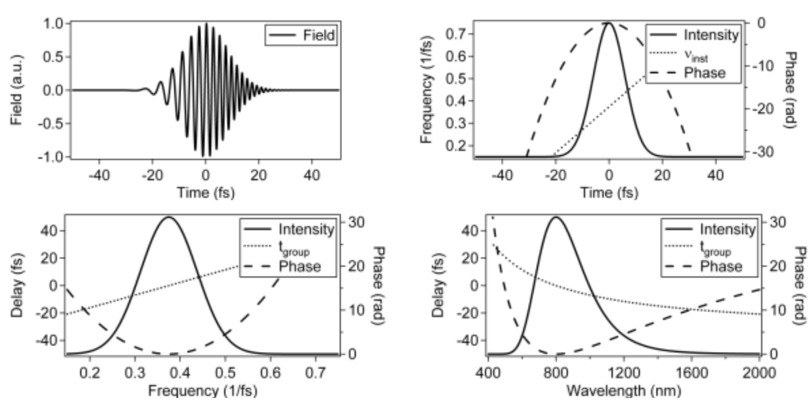
2^{nd}-order phase: positive linear chirp

•Numerical example: Gaussian-intensity pulse w/ positive linear chirp, $\phi_2 = -0.032$ rad/fs^2 or $\varphi_2 = 290$ rad fs^2.



Here the quadratic phase has stretched what would have been a 3-fs pulse (given the spectrum) to a 13.9-fs one.

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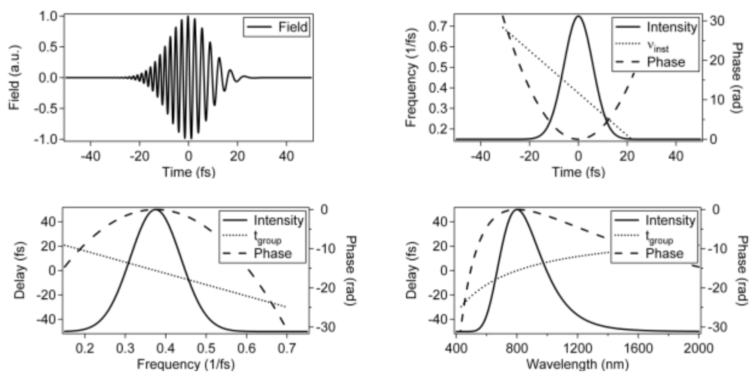
2nd-order phase: negative linear chirp

•Numerical example: Gaussian-intensity pulse w/ negative linear chirp, ϕ_2 =+0.032 rad/fs^2 or $\varphi_2 = -290$ rad fs^2.



As with positive chirp, the quadratic phase has stretched what would have been a 3-fs pulse (given the spectrum) to a 13.9-fs one.

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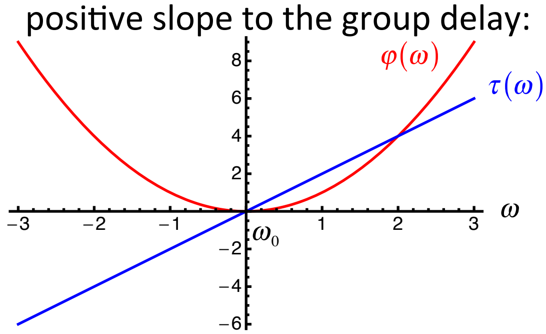
Group delay vs spectral phase

- The group delay gives the arrival time of the different frequency components

$$\tau_g(\omega) = \frac{d\varphi}{d\omega}$$

$$\varphi(\omega) = \varphi_0 + \varphi_1 \frac{\omega-\omega_0}{1!} + \varphi_2 \frac{(\omega-\omega_0)^2}{2!} + \ldots$$

- So a positive 2nd order phase gives a positive slope to the group delay:



Not usually important:

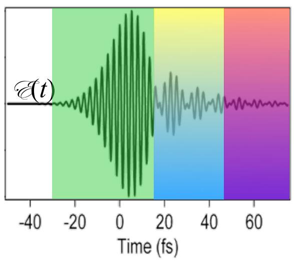
- phase constant
- group delay shift

Use group delay variation to visualize chirp.

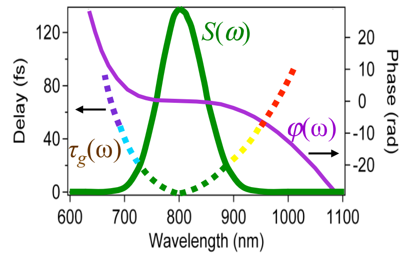
3rd-order spectral phase: quadratic chirp

•The red and blue colors coincide in time and interfere.

E-field vs. time



Spectrum and spectral phase



Trailing satellite pulses in time indicate positive spectral cubic phase, and leading ones indicate negative spectral cubic phase.

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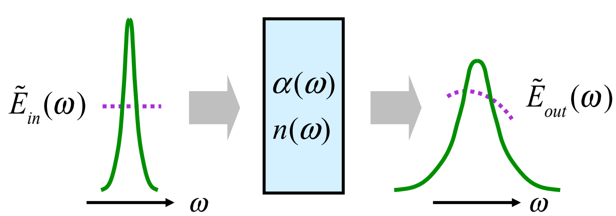
Pulse propagation

•What happens to a pulse as it propagates through a medium?

•Always model (linear) propagation in the **frequency domain**. Also, you must know the entire field (i.e., the intensity and phase) to do so.



$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega) \exp\left[-\frac{\alpha(\omega)}{2}L\right]\exp\left[i\,k(\omega)\,L\right]$$

In the time domain, propagation is a convolution—much harder.

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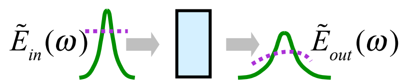
Pulse propagation (continued)



Rewriting this expression using $k = n(\omega)\,\omega/c$:

$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega)\exp[-\alpha(\omega)L/2]\exp[i\omega n(\omega)\,L/c]$$

Separating out the spectrum and spectral phase:

$$S_{out}(\omega) = S_{in}(\omega)\exp[-\alpha(\omega)L]$$

$$\varphi_{out}(\omega) = \varphi_{in}(\omega) + n(\omega)\frac{\omega}{c}L$$

Absorption (or gain) modifies the spectral amplitude,
Refractive index modifies the spectral phase

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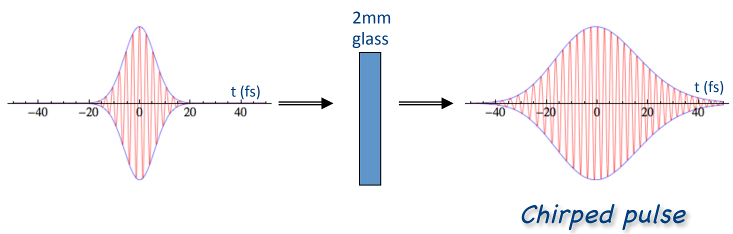
Pulse propagation: t/ω domains

- Dispersion in a system will stretch a short pulse:

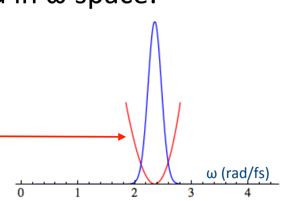


- Linear propagation is best represented in ω space:

$$E_{out}(\omega) = A(\omega - \omega_0)e^{i\phi(\omega)}$$

Spectral phase

$$\phi(\omega) = k\,L = \frac{\omega}{c}n(\omega)L$$



Propagation of a Gaussian pulse

- Start with pulse in t-domain

$$E(z=0,t)=A_0\, e^{-t^2/t_0^2}e^{-i\,\omega_0\, t}$$

- FT to frequency space:

$$E(z=0,\omega)=FT\{E(t)\}=A_0\, t_0\, e^{-\frac{1}{4}(\omega-\omega_0)^2\, t_0^2}$$

- Apply phase shift that results from propagation:

$$E(z,\omega)=A_0\, t_0\, e^{-\frac{1}{4}(\omega-\omega_0)^2\, t_0^2}\, e^{i\frac{\omega}{c}n(\omega)z}\approx A_0\, t_0\, e^{-\frac{1}{4}(\omega-\omega_0)^2\, t_0^2}\, e^{i\left(\varphi_0+(\omega-\omega_0)\varphi_1+\frac{1}{2}(\omega-\omega_0)^2\varphi_2\right)}$$

$$=A_0\, t_0\, e^{i\varphi_0}\exp\left[i(\omega-\omega_0)\varphi_1\right]\exp\left[-(\omega-\omega_0)^2\left(\frac{t_0^2}{4}-i\frac{1}{2}\varphi_2\right)\right]$$

Constant phase   Group delay shift   Chirp

  - Note that the phase terms are typically proportional to z

- Next: inverse transform to t-domain.

Propagated pulse in time domain

- In the time-domain, pulse can be written

$$E(z,t)=A_0\, t_0\frac{1}{2\pi}\int e^{i\varphi_0}\exp\left[i(\omega-\omega_0)\varphi_1\right]\exp\left[-(\omega-\omega_0)^2\left(\frac{t_0^2}{4}-i\frac{1}{2}\varphi_2\right)\right]e^{-i\omega t}\, d\omega$$

- We will use the shift theorem for carrier and group delay, so consider this integral:

$$f(t)=\frac{1}{2\pi}\int \exp\left[-\delta\omega^2\left(\frac{t_0^2}{4}-i\frac{1}{2}\varphi_2\right)\right]e^{-i\,\delta\omega\, t}\, d\delta\omega$$

- So that

$$E(z,t)=A_0\, t_0 e^{i\,\varphi_0-i\omega_0 t}f(t-\varphi_1)$$

- Note that the group delay is just the transit time through

$$\varphi_1=\tau_g(\omega_0)=\left.\frac{d\varphi}{d\omega}\right|_{\omega=\omega_0}=\left.\frac{dk}{d\omega}\right|_{\omega=\omega_0}\cdot L=\frac{L}{\mathrm{v}_g}$$

Chirped output pulse

- We're doing the FT of a complex Gaussian

$$f(t)=\frac{1}{2\pi}\int \exp\left[-\delta\omega^2\left(\frac{t_0^2}{4}-i\frac{1}{2}\varphi_2\right)\right]e^{-i\,\delta\omega\,t}\,d\delta\omega$$

$$FT^{-1}\left\{\exp\left(-T^2\omega^2/4\right)\right\}=\frac{1}{\sqrt{\pi\,T^2}}\exp\left(-t^2/T^2\right)$$

$$T^2=t_0{}^2-2i\varphi_2$$

T^2 = "complex time"
~ q(z) for Gaussian beams

$$f(t)=\frac{1}{\sqrt{\pi\left(t_0^2-2i\varphi_2\right)}}\exp\left(-\frac{t^2}{t_0^2-2i\varphi_2}\right)$$

$$\frac{1}{t_0^2-2i\varphi_2}=\frac{t_0^2+2i\varphi_2}{t_0^4+4\varphi_2^2}=\frac{1+\frac{2i\varphi_2}{t_0^2}}{t_0^2\left(1+\left(\frac{2\varphi_2}{t_0^2}\right)^2\right)}=\frac{1+\frac{2i\varphi_2}{t_0^2}}{\tau^2(z)}$$

Chirped output pulse

- The pulse duration and chirp parameter vary with z

z-dependent pulse duration

$$\tau(z)=t_0\sqrt{1+\left(\frac{2\varphi_2}{t_0^2}\right)^2}=t_0\sqrt{1+\left(\frac{2k_2}{t_0^2}z\right)^2}$$

$$\varphi_2(z)=\left.\frac{d^2\varphi}{d\omega^2}\right|_{\omega=\omega_0}=z\left.\frac{d^2k}{d\omega^2}\right|_{\omega=\omega_0}=k_2z$$

z-dependent chirp parameter

$$\beta(z)=\frac{1}{\tau^2(z)}\frac{2\varphi_2}{t_0^2}$$

$$f(t)=\frac{1}{\sqrt{\pi\left(t_0^2-2i\varphi_2\right)}}\exp\left(-\frac{t^2}{\tau^2(z)}\right)\exp\left(-i\beta t^2\right)$$

- This dispersion dependence is just like a Gaussian beam that focuses and diverges.

Final form of E(z,t)

- Leading factor:

$$f(t)=\frac{1}{\sqrt{\pi\left(t_0^2-2i\varphi_2\right)}}\exp\left(-\frac{t^2}{\tau^2(z)}\right)\exp\left(-i\beta t^2\right) \qquad \tau(z)=t_0\sqrt{1+\left(\frac{2\varphi_2}{t_0^2}\right)^2}$$

$$\frac{1}{\sqrt{t_0{}^2-2i\varphi_2}}=\sqrt{\frac{1+\frac{2i\varphi_2}{t_0{}^2}}{\tau^2(z)}}=\frac{1}{\tau(z)}\sqrt{\left(1+\frac{4\varphi_2{}^2}{t_0{}^4}\right)^{1/2}\exp\left[i\arctan\left(\frac{2\varphi_2}{t_0{}^2}\right)\right]}$$

$$=\frac{1}{t_0\left(1+\frac{4\varphi_2{}^2}{t_0{}^4}\right)^{1/4}}\exp\left[\frac{i}{2}\arctan\left(\frac{2\varphi_2}{t_0{}^2}\right)\right]$$

$$f(t)=\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{t_0\tau(z)}}\exp\left[\frac{i}{2}\arctan\left(\frac{2\varphi_2}{t_0{}^2}\right)\right]\exp\left(-\frac{t^2}{\tau^2(z)}\right)\exp\left(-i\beta t^2\right)$$

Final form of E(z,t)

- Complete form of Gaussian pulse propagation

$$E(z,t)=\frac{A_0}{\sqrt{\pi}}\frac{1}{\sqrt{t_0\tau(z)}}e^{-i\omega_0 t+i\varphi_0}\,e^{\frac{i}{2}\arctan\left(\frac{2\varphi_2}{t_0{}^2}\right)}\exp\left(-\frac{(t-\varphi_1)^2}{\tau^2(z)}-i\beta(t-\varphi_1)^2\right)$$

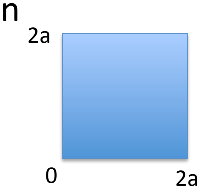
- Intensity follows 1/pulse duration
- z-dependent phase term similar to the spatial Gouy phase
- Pulse envelope moves at the group velocity
- Dispersion length: characteristic distance for stretching:

$$\tau(z)=t_0\sqrt{1+\left(\frac{2k_2}{t_0{}^2}z\right)^2} \qquad L_d=\frac{t_0{}^2}{2k_2}$$

τ increases by sqrt(2) over distance L_d

Modal dispersion

- Confinement of the propagating mode gives a geometric contribution to the dispersion
- Example: square waveguide



$$\nabla^2 E + n^2 \frac{\omega^2}{c^2} E = 0$$

$$\rightarrow n^2 \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2$$

- Find transverse modes: $E(x,y,z) = E_0 \sin(k_x x)\sin(k_y y)e^{ik_z z}$

$$k_x \cdot 2a = m_x \pi \qquad k_x = \frac{m_x \pi}{2a} \quad k_y = \frac{m_y \pi}{2a}$$

Indices ≥ 1

$$\rightarrow k_z(\omega) = \sqrt{n^2 \frac{\omega^2}{c^2} - k_x^2 - k_y^2} = \sqrt{n^2 \frac{\omega^2}{c^2} - \frac{\pi^2}{4a^2}\left(m_x^2 + m_y^2\right)}$$

Dispersion depends on mode

Modal dispersion affects phase and group velocity

- Group delay dispersion has a geometric contribution
- Consider simple case: vacuum-filled hollow waveguide

$$k_z(\omega) = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{4a^2}\left(m_x^2 + m_y^2\right)}$$

$$v_{ph} = \frac{\omega_0}{k} = \frac{c}{\sqrt{1 - \frac{\pi^2 c^2}{4a^2\omega^2}\left(m_x^2 + m_y^2\right)}}$$

Faster phase velocity

$$k_1 = \left.\frac{\partial k_z}{\partial \omega}\right|_{\omega_0} = \frac{1}{c\sqrt{1 - \frac{\pi^2 c^2}{4a^2\omega_0^2}\left(m_x^2 + m_y^2\right)}}$$

$$v_{gr} = \left.\frac{\partial \omega}{\partial k_z}\right|_{\omega_0} = c\sqrt{1 - \frac{\pi^2 c^2}{4a^2\omega^2}\left(m_x^2 + m_y^2\right)}$$

Slower group velocity

Waveguide dispersion: GDD

- The second-order phase is negative

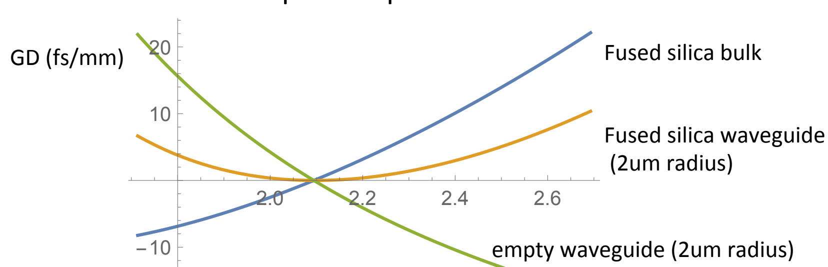
$$k_2 = \left.\frac{\partial^2 k_z}{\partial \omega^2}\right|_{\omega_0} = \frac{1}{\omega c\sqrt{1-\frac{\pi^2 c^2}{4a^2\omega_0^2}\left(m_x^2+m_y^2\right)}} - \frac{1}{\omega c\sqrt{1-\frac{\pi^2 c^2}{4a^2\omega_0^2}\left(m_x^2+m_y^2\right)}}$$

$$k_2 = \left.\frac{\partial^2 k_z}{\partial \omega^2}\right|_{\omega_0} = -\frac{k_1}{\omega}\left(\frac{c^2}{v_g^{\,2}}-1\right)$$

Group velocity < c

Balancing material and waveguide dispersion

- Mix of positive (material) and negative (waveguide) GDD leads to a zero-dispersion point



- Standard single-mode fiber (SMF): ZDP ~ 1500nm
- Photonic crystal fiber (PCF): small core size to push ZDP to lower wavelengths