**MATH 348 - Advanced Engineering Mathematics** March 24, 2008
**Homework 7, Spring 2008** **Due**: April 2, 2008

PARTIAL DIFFERENTIAL EQUATIONS - THE ONE-DIMENSIONAL HEAT EQUATION - GENERALIZATIONS

Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{1}$$

$$x \in (0, L), \quad t \in (0, \infty), \quad c^2 = \frac{K}{\sigma\rho}. \tag{2}$$

Equations (1)-(2) model the time-evolution of the temperature, $u = u(x,t)$, of a heat conducting medium in one-dimension. The object, of length $L$, is assumed to have a homogenous thermal conductivity $K$, specific heat $\sigma$, and linear density $\rho$. That is, $K, \sigma, \rho \in \mathbb{R}^+$.

1. Consider the one-dimensional heat equation, (1)-(2), with the boundary conditions[1],

$$u_x(0,t) = 0, u(L,t) = 0, \tag{3}$$

and initial condition

$$u(x,0) = f(x). \tag{4}$$

   (a) Assume that the solution to (1)-(2) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (3)-(4). [2]
   (b) What is the temperature in the medium for when $t$ is large? That is, determine $\lim_{t\to\infty} u(x,t)$.

2. Consider the one-dimensional heat equation, (1)-(2), with the boundary conditions[3],

$$u_x(0,t) = 0, u_x(L,t) = 0, \tag{5}$$

$$u(x,0) = f(x). \tag{6}$$

   (a) Assume that the solution to (1)-(2) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (5)-(6). [4]
   (b) Describe how the long term behavior of the general solution to (1)-(6) changes as the thermal conductivity, $K$, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, $\rho$, is increased while all other parameters are held constant.

Define,

$$f(x) = \begin{cases} x, & 0 < x \leq L \\ -x + 2L, & L < x < 2L \end{cases} \tag{7}$$

3. For the following questions we consider the solution, $u$, to the heat equation given by, (1)-(2), which satisfies the initial condition given by (7). [5]

   (a) Suppose this solution also satisfies the boundary conditions are given by $u(0,t) = u(2L,t) = 0$. Determine the unique solution the PDE satisfying all of these conditions. What is $\lim_{t\to\infty} u(x,t)$?
   (b) For $L = 2L$, find the particular solution to (1)-(2) with boundary conditions (5)-(6) for when the initial temperature profile of the medium is given by (7). Show that $\lim_{t\to\infty} u(x,t) = f_{avg} = .25$.[6]

[1] Here the boundary conditions correspond to making the left endpoint perfectly insulated while the right endpoint is fixed at zero degrees

[2] This is done for different boundary conditions in Kreyszig pp.552-556.

[3] Here the boundary conditions correspond to perfect insulation of both endpoints

[4] An insulated bar is discussed in examples 4 and 5 on page 557.

[5] When solving for the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.

[6] It is interesting here to note that though the initial condition $f$ doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function $f$ at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

4. Recall the 1-D conservation law encountered during the derivation of the heat equation.[7]

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x}, \; \kappa \in \mathbb{R} \tag{8}$$

In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the $\phi$ is proportional to the negative gradient of $u$ then, from (8), we get the one-dimensional heat/diffusion equation (1).[8]

(a) Assume that $\phi$ is proportional to $u$ to derive, from (8), the convection/transport equation, $u_t + cu_x = 0$, where $c$ is some proportionality constant.

(b) Given the initial condition $u(x,0) = u_0(x)$ for the convection equation, show that $u(x,t) = u_0(x-ct)$ is a solution to this PDE.

(c) If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (8), the convection-diffusion equation $u_t + cu_x - du_{xx} = 0$.

(d) If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term $\lambda u$ to get the convection-diffusion-decay equation,[9]

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{9}$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (9) can be transformed into a heat equation on the new variable $w$ where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[10]

5. Consider `http://en.wikipedia.org/wiki/Heat_transfer` and respond to the the following:

(a) Define conduction, convection, and radiation.

(b) Thermodynamically speaking, why must heat transfer always occur from hot to cold bodies?

(c) Give examples of three disciplines involving heat transfer methods.

[7] When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity $u$ could be charge density and $q$ would be its flux.

[8] AKA Fick's Second Law associated with linear non-steady-state diffusion.

[9] The $u_{xx}$ term models diffusion of energy/particles while $u_x$ models convection, $u$ models energy/particle loss/decay.

[10] This shows that the general PDE (9) can be solved using heat equation techniques.