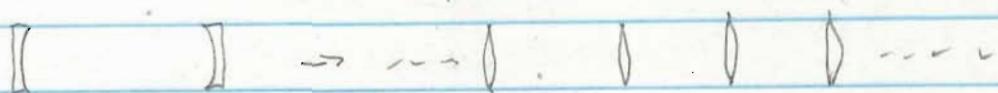


Stability of resonators.

Ray optics approach:

a mirror resonator can be unfolded \rightarrow periodic lens array



are there rays that stay bounded within resonator?
n round trips:

$$\begin{pmatrix} r_n \\ r'_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \begin{pmatrix} r_0 \\ r'_0 \end{pmatrix}$$

matrix is for one round-trip.

each ray is not necessarily the same, but $|r_n|, |r'_n| < \infty$
 \rightarrow condition on matrix

let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = M$ diagonalize $\rightarrow M = U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1}$

then $M^n = U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1} U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1} \dots$

$$= U \begin{pmatrix} \lambda_a^n & 0 \\ 0 & \lambda_b^n \end{pmatrix} U^{-1} \quad \text{since } U^{-1}U = I$$

require $|\lambda_a| \leq 1$ $|\lambda_b| \leq 1$ for stability

reverse propagation:

$$|\lambda_a| \leq 1 \quad |\lambda_b| \leq 1$$

\therefore matrix must satisfy: $|\lambda_a| = |\lambda_b| = 1$

$$\det M = \lambda_a \lambda_b = 1 \rightarrow \lambda_a = e^{i\theta} \quad \lambda_b = \lambda_a^* = e^{-i\theta}$$

$$\text{Tr } M = \lambda_a + \lambda_b = A + D = e^{i\theta} + e^{-i\theta} = 2 \cos \theta$$

$\boxed{-1 \leq \frac{A+D}{2} \leq 1}$ stability condition

Comments on ABCD matrix properties:

some confusion in text about unitarity.

1) determinant = 1 if start and end one is same medium (same index)

ex. lens $\begin{pmatrix} 1 & 0 \\ -1/\mu & 1 \end{pmatrix}$, prop $\begin{pmatrix} 1 & 4/n \\ 0 & 1 \end{pmatrix}$

counter ex. interface: $\begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$

* $\det M=1 \rightarrow \lambda_a \lambda_b = 1$ but λ 's can be real or complex!

2) complex eigenvalues of form $e^{\pm i\theta}$

→ stable cavity $\lambda_a \lambda_b = 1$, $\lambda_a + \lambda_b = 2 \cos \theta$

outside this range λ 's are real

$$\lambda_a = 1/\lambda_b$$

$$\text{Tr}(M) = \lambda_a + 1/\lambda_a$$

$$\text{if } \lambda_a = 1 \rightarrow \text{Tr}(M) = 2$$

$$\lambda_a > 1 \rightarrow \text{Tr}(M) > 2$$

* if $\lambda = e^{\pm i\theta}$ M is not necessarily unitary.

unitary means $M^{-1} = M^*$ conjugate transpose

Gaussian beam stability

- here, we require field repeats on every round-trip.

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} \quad q_1 = q_0$$

$$\Rightarrow q_0(Cq_0 + D) = Aq_0 + B$$

$$Cq_0^2 + (D-A)q_0 - B = 0$$

$$\text{solve } -\frac{(D-A)}{2C} \pm \frac{1}{2C} \sqrt{(D-A)^2 + 4BC}$$

q must be complex, or $(D-A)^2 + 4BC < 0$

also ABCD matrix has property $\det m = 1 = AD - BC$

$$\Rightarrow D^2 + A^2 - 2AD + 4(AD - 1) = D^2 + A^2 + 2AD - 4 < 0$$

$$\frac{(A+D)^2}{4} < 1 \quad \text{same condition.}$$

Physical interpretation:

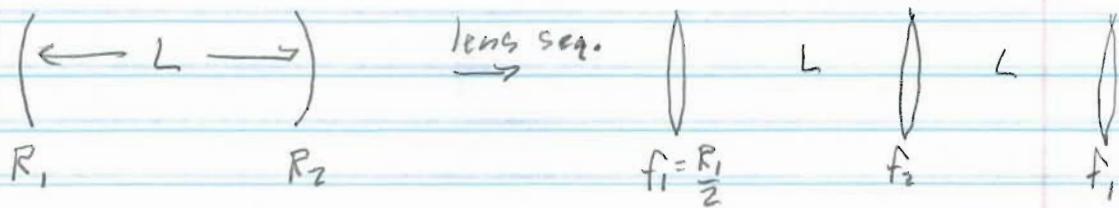
mode R = end mirror R

choose mode size, propagate out to mirror location,

pick R_{mirror} .

Stability of a 2 mirror cavity.

- cavities with more mirrors can often be mapped onto this model



$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & L \\ -\frac{1}{f_1} & 1 - \frac{L}{f_1} \end{pmatrix} \begin{pmatrix} 1 & L \\ -\frac{1}{f_2} & 1 - \frac{L}{f_2} \end{pmatrix}$$

evaluate stability: $(A+D)/2$

$$A = 1 - \frac{L}{f_2}$$

$$D = -\frac{L}{f_1} + \left(1 - \frac{L}{f_1}\right) \left(1 - \frac{L}{f_2}\right)$$

$$\frac{A+D}{2} = \frac{1}{2} \left[1 - \frac{L}{f_2} - \frac{L}{f_1} + 1 - \frac{L}{f_1} - \frac{L}{f_2} + \frac{L^2}{f_1 f_2} \right]$$

$$= 1 - \frac{L}{f_1} - \frac{L}{f_2} + \frac{L^2}{2f_1 f_2}$$

change to R 's, can write as

$$\frac{A+D}{2} = 2 \left(1 - \frac{L}{R_1}\right) \left(1 - \frac{L}{R_2}\right) - 1 = 2g_1 g_2 - 1$$

stable if $-1 \leq 2g_1 g_2 - 1 \leq 1$

or

$$0 \leq g_1 g_2 \leq 1$$

Stability map



shaded = stable

$$g_1 = 1 - \frac{L}{R_1}$$

$$g_2 = 1 - \frac{L}{R_2}$$

concentric

$$L = 2R$$

$$g_1 = g_2 = -1$$

confocal

$$L = R$$

$$g_1 = g_2 = 0$$

planar

$$R = \infty$$

$$g_1 = g_2 = +1$$

Finding waist location + size:

if end mirror is flat \rightarrow waist.

also have waist between two curved mirrors.

e.g.



find q at curved mirror.

propagate forward to waist (where $Re(q) \rightarrow 0$)

$$\text{given } q_0 \rightarrow z_m = n \frac{\text{Re}[q_0]}{|q_0|^2}$$

$$z_{R0} = z_0 \frac{R_i^2}{R_i^2 + z_i^2} \rightarrow w_0,$$

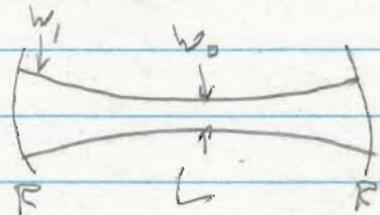
see Gively for specific formulas.

Symmetric resonators: w_0 is in center of cavity

mirror R follows Gaussian beam:

$$R(z) = z \left(1 + \frac{z_0^2}{z^2}\right) \quad z \rightarrow L/2$$

$$R = \frac{L}{2} \left(1 + \frac{4z_0^2}{L^2}\right)$$



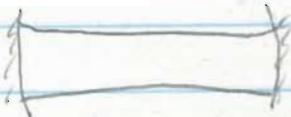
confocal $L = R$

$$\Rightarrow 2 = 1 + \frac{4z_0^2}{R^2} \Rightarrow z_0^2 = \frac{R^2}{4} \quad \frac{\pi w_0^2}{\lambda} = \frac{R}{2}$$

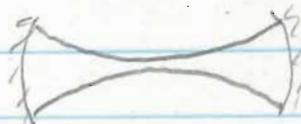
$$w_0 = \left(\frac{R\lambda}{2\pi}\right)^{1/2} \text{ at center}$$

$$\text{also note } \frac{L}{2} = R = z_0 \text{ so } w_1 = \sqrt{2} w_0$$

near planar $W_0 \approx W_1$



near concentric \rightarrow small spot.



As position on stability map changes \rightarrow different mode sizes.

Finding stable mode cases

- if cavity M is stable \rightarrow one value of q that corresponds to mode
- other values will repeat, but not on one round trip.

eigenmode:

$$q = \frac{Aq + B}{Cq + D} \quad \text{as before}$$

$$\rightarrow q = -\frac{(D-A)}{2C} \pm \frac{1}{2C} \sqrt{(D-A)^2 + 4BC}$$

$$\text{since } \det M = 1 = AD - BC \rightarrow BC = AD - 1$$

$$D^2 + A^2 - 2AD + 4AD - 4 = (A+D)^2 - 4 < 1$$

The value of q obtained is at the starting point w/o cavity.
If starting point is at a flat mirror, $q = \text{imaginary}$.

$\therefore D = A \rightarrow q = \pm \sqrt{\frac{B}{C}}$ which is imaginary, since stability requires $\sqrt{-\text{complex}}$

$$\text{since } \frac{1}{q} = \frac{1}{R} - \frac{i}{\lambda}$$

$$\text{with } R = \infty \quad q = i\frac{\lambda}{2} = i\frac{\pi w^2}{\lambda} \quad \text{take +ve root.}$$

If we start at end of cavity, we can find mode by changing ABCD



$$f = R/2$$



\rightarrow



$f = R$ b/c double pass

$$\frac{1}{f_2} = \frac{1}{f_1} + \frac{1}{A_1}$$

$$2/R = \frac{1}{f_1} + \frac{1}{R}$$

Example: compare a near-planar resonator w/ confocal
symmetric cavity $R_1 = R_2 = R$
separation $L \ll R$



What is spot size at end mirror and waist?

$$R(z) = z \left(1 + \left(\frac{z_R}{z} \right)^2 \right)$$

$z = \text{dist to waist}$
 $= L/2$

$$R = \frac{L}{2} \left(1 + 4 \frac{z_R^2}{L^2} \right)$$

$$z_R = \frac{L}{2} \left(2R - 1 \right)^{1/2}$$

$R \gg L \text{ so } z_R \approx \sqrt{\frac{RL}{2}}$

and $w_o = \left(\frac{\lambda z_R}{\pi} \right)^{1/2}$

for $L = 1\text{m}$ $R = 8\text{m} \rightarrow w_o \approx 0.57\text{ mm}$

$\lambda = 514\text{ nm}$ (Ar⁺ laser)

At end mirror?

$$w(z) = w_o \sqrt{1 + \left(\frac{z}{z_R} \right)^2}$$

$$= w_o \sqrt{1 + \left(\frac{L}{2\sqrt{RL}} \right)^2} = w_o \sqrt{1 + \frac{L}{2R}}$$

In this case $w_i = w_o \cdot (1^{1/2})$ \sim constant mode size.

Compare to confocal of same L:

$R_1 = R_2 = L = 1\text{m}$

$$z_R = \frac{L}{2} \left(2 - 1 \right)^{1/2} = \frac{L}{2} \rightarrow w_o = \left(\frac{\lambda L}{2\pi} \right)^{1/2} = 0.29\text{ mm}$$

$w_i = \sqrt{2} w_o = 0.4\text{ mm}$

Higher-order modes.

Gaussian beam is just lowest order solution.

general case \rightarrow Laguerre-Gaussian (cylindrical coords)
Hermite-Gaussian (cartesian coords)

Hermite-Gaussian.

$$U_{lm}(x, y, z) = \frac{w}{w_0} H_l\left(\frac{\sqrt{2}x}{w}\right) H_m\left(\frac{\sqrt{2}y}{w}\right) e^{-\frac{(x^2+y^2)}{w^2}} e^{-ik\frac{(x^2+y^2)}{2R} + i(l+m)\phi}$$

$w(z)$, $R(z)$, $\phi(z)$ as before.

$H_l(s)$ = Hermite polynomials.

solutions are similar to quantum 2D

Notes $l, m = 0$ = TEM₀₀ = Gaussian

l, m correspond to number of nodes

$l, m > 0 \rightarrow$ wider extent of field.

pattern of mode is invariant with z

- only scale changes as $w(z)$ increases

If system is azimuthally symmetric \rightarrow degeneracy.

e.g. $1, 0$ same as $0, 1$



Radius of curvature same for all modes w/ same w_0

eigenvalues of cavities \rightarrow mode structure.

again Fabry-Pérot, require RT phase = integer $\cdot 2\pi$
 \rightarrow resonance frequencies.

$$\sqrt{\lambda_{mn}} = \frac{c}{2L} \left[n + \frac{1+l+m}{\pi} \cos^{-1}(\pm \sqrt{g_1 g_2}) \right] \quad \begin{matrix} 2 \text{ mirror} \\ \text{cavity.} \end{matrix}$$

For given l, m mode spacing ($\Delta\lambda = \sqrt{\lambda_{m,n+1}} - \sqrt{\lambda_{m,n}}$)
 $\Delta\lambda = \frac{c}{2L} \cancel{g_1 g_2}$.

Frequencies are shifted for different l, m .

Mode selection + finite aperture.

aperture \rightarrow diffraction losses.

aperture can select mode: more loss for $l, m > 0$