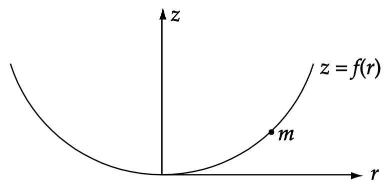
10-6.



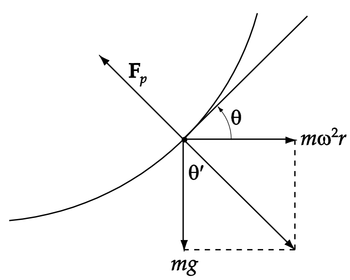
Consider a small mass m on the surface of the water. From Eq. (10.25)

$$\mathbf{F}_{\text{eff}} = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times r) - 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

In the rotating frame, the mass is at rest; thus, $\mathbf{F}_{\text{eff}} = 0$. The force $\mathbf{F}$ will consist of gravity and the force due to the pressure gradient, which is normal to the surface in equilibrium. Since $\ddot{\mathbf{R}}_f = \dot{\boldsymbol{\omega}} = \mathbf{v}_r = 0$, we now have

$$0 = m\mathbf{g} + \mathbf{F}_p - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{F}_p$ is due to the pressure gradient.



Since $\mathbf{F}_{\text{eff}} = 0$, the sum of the gravitational and centrifugal forces must also be normal to the surface.

Thus $\theta' = \theta$.

$$\tan\theta' = \tan\theta = \frac{\omega^2 r}{g}$$

but

$$\tan\theta = \frac{dz}{dr}$$

Thus

$$\boxed{z = \frac{\omega^2}{2g} r^2 + \text{constant}}$$

The shape is a circular paraboloid.

10-7. For a spherical Earth, the difference in the gravitational field strength between the poles and the equator is only the centrifugal term:

$$g_{\text{poles}} - g_{\text{equator}} = \omega^2 R$$

For $\omega = 7.3 \times 10^{-5} \text{rad} \cdot \text{s}^{-1}$ and R = 6370 km, this difference is only $34 \text{ mm} \cdot \text{s}^{-2}$. The disagreement with the true result can be explained by the fact that the Earth is really an oblate spheroid, another consequence of rotation. To qualitatively describe this effect, approximate the real Earth as a somewhat smaller sphere with a massive belt about the equator. It can be shown with more detailed analysis that the belt pulls inward at the poles more than it does at the equator. The next level of analysis for the undaunted is the "quadrupole" correction to the gravitational potential of the Earth, which is beyond the scope of the text.

10-9. Choosing the same coordinate system as in Example 10.3 (see Fig. 10-9), we see that the lateral deflection of the projectile is in the x direction and that the acceleration is

$$a_x = \ddot{x} = 2\,\omega_z\, v_y = 2(\omega \sin\lambda)(V_0 \cos\alpha) \tag{1}$$

Integrating this expression twice and using the initial conditions, $\dot{x}(0) = 0$ and $x(0) = 0$, we obtain

$$x(t) = \omega V_0 t^2 \cos\alpha \sin\lambda \tag{2}$$

Now, we treat the z motion of the projectile as if it were undisturbed by the Coriolis force. In this approximation, we have

$$z(t) = V_0 t \sin\alpha - \frac{1}{2} g t^2 \tag{3}$$

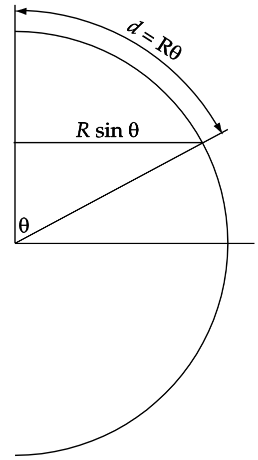
from which the time T of impact is obtained by setting z = 0:

$$T = \frac{2V_0 \sin\alpha}{g} \tag{4}$$

Substituting this value for T into (2), we find the lateral deflection at impact to be

$$\boxed{x(T) = \frac{4\omega V_0^3}{g^2} \sin\lambda \cos\alpha \sin^2\alpha} \tag{5}$$

10-11.



This problem is most easily done in the fixed frame, not the rotating frame. Here we take the Earth to be fixed in space but rotating about its axis. The missile is fired from the North Pole at some point on the Earth's surface, a direction that will always be due south. As the missile travels towards its intended destination, the Earth will rotate underneath it, thus causing it to miss. This distance is:

$$\Delta = (\text{transverse velocity of Earth at current latitude}) \times (\text{missile's time of flight})$$

$$= \omega R \sin\theta \times T \tag{1}$$

$$= \frac{d\omega R}{v} \sin\left(\frac{d}{R}\right) \tag{2}$$

Note that the actual distance d traveled by the missile (that distance measured in the fixed frame) is less than the flight distance one would measure from the Earth. The error this causes in Δ will be small as long as the miss distance is small. Using R = 6370 km, $\omega = 7.27 \times 10^{-5}$ rad $\cdot\, s^{-1}$, we obtain for the 4800 km, T = 600 s flight a miss distance of 190 km. For a 19300 km flight the missile misses by only 125 km because there isn't enough Earth to get around, or rather there is less of the Earth to miss. For a fixed velocity, the miss distance actually peaks somewhere around d = 12900 km.

Doing this problem in the rotating frame is tricky because the missile is constrained to be in a path that lies close to the Earth. Although a perturbative treatment would yield an order of magnitude estimate on the first part, it is entirely wrong on the second part. Correct treatment in the rotating frame would at minimum require numerical methods.

10-15. The Lagrangian in the fixed frame is

$$L = \frac{1}{2} m v_f^2 - U\left(r_f\right) \tag{1}$$

where v_f and r_f are the velocity and the position, respectively, in the fixed frame. Assuming we have common origins, we have the following relation

$$\mathbf{v}_f = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}_r \tag{2}$$

where v_r and r_r are measured in the rotating frame. The Lagrangian becomes

$$L = \frac{m}{2}\left[v_r^2 + 2\mathbf{v}_r \cdot (\boldsymbol{\omega} \times \mathbf{r}_r) + (\boldsymbol{\omega} \times \mathbf{r}_r)^2\right] - U(r_r) \tag{3}$$

The canonical momentum is

$$\mathbf{p}_r \equiv \frac{\partial L}{\partial \mathbf{v}_r} = m\mathbf{v}_r + m(\boldsymbol{\omega} \times \mathbf{r}_r) \tag{4}$$

The Hamiltonian is then

$$H \equiv \mathbf{v}_r \cdot \mathbf{p}_r - L = \frac{1}{2}mv_r^2 - U(r_r) - \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r}_r)^2 \tag{5}$$

H is a constant of the motion since $\partial L/\partial t = 0$, but $H \neq E$ since the coordinate transformation equations depend on time (see Section 7.9). We can identify

$$U_c = -\frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r}_r)^2 \tag{6}$$

as the centrifugal potential energy because we may find, with the use of some vector identities,

$$-\nabla U_c = \frac{m}{2}\nabla\left[\omega^2 r_r^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_r)^2\right] \tag{7}$$

$$= m\left[\omega^2 \mathbf{r}_r - (\boldsymbol{\omega} \cdot \mathbf{r}_r)\boldsymbol{\omega}\right] \tag{8}$$

$$= -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \cdot \mathbf{r}_r) \tag{9}$$

which is the centrifugal force. Computing the derivatives of (3) required in Lagrange's equations

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \mathbf{v}_r} = m\mathbf{a}_r + m\boldsymbol{\omega} \times \mathbf{v}_r \tag{10}$$

$$\frac{\partial \mathbf{L}}{\partial \mathbf{r}_r} = m\nabla\left[(\mathbf{v}_r \times \boldsymbol{\omega}) \cdot \mathbf{r}_r\right] - \nabla(U_c + U) \tag{11}$$

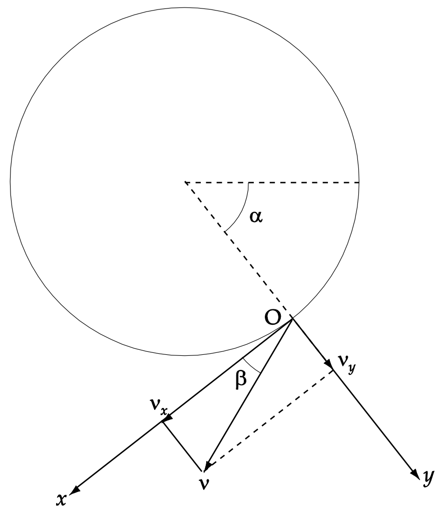
$$= -m(\boldsymbol{\omega} \times \mathbf{v}_r) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_r) - \nabla U \tag{12}$$

The equation of motion we obtain is then

$$m\mathbf{a}_r = -\nabla U - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_r) - 2m(\boldsymbol{\omega} \times \mathbf{v}_r) \tag{13}$$

If we identify $\mathbf{F}_{\text{eff}} = m\mathbf{a}_r$ and $\mathbf{F} = -\nabla U$, then we do indeed reproduce the equations of motion given in Equation 10.25, without the second and third terms.

10-18. Let us choose the coordinate system Oxyz as shown in the figure.



The projectile's velocity is

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} = \begin{pmatrix} v_0 \cos\beta \\ v_0 \sin\beta - gt \\ 0 \end{pmatrix} \quad \text{where } \beta = 37°$$

The Earth's angular velocity is

$$\vec{\omega} = \begin{pmatrix} -\omega\cos\alpha \\ -\omega\sin\alpha \\ 0 \end{pmatrix} \quad \text{where} \quad \alpha = 50°$$

So the Coriolis acceleration is

$$\vec{a}_c = 2\vec{v}\times\vec{\omega} = \left(-2v_0\ \omega\cos\beta\sin\alpha + 2\left(v_0\sin\beta - gt\right)\omega\cos\alpha\right)\mathbf{e}_z$$

The velocity generated by Coriolis force is

$$v_c = \int_0^t a_c\ dt = 2v_0\ \omega t\left(\cos\beta\sin\alpha - \sin\beta\cos\alpha\right) - gt^2\omega\cos\alpha$$

And the distance of deviation due to the Coriolis force is

$$z_c = \int_0^t v_c\ dt = -v_0\ \omega t^2\ \sin\left(\alpha-\beta\right) - \frac{gt^3\omega\cos\alpha}{3}$$

The flight time of the projectile is $t = \dfrac{2v_0\sin\beta}{2}$. If we put this into z_c, we find the deviation distance due to Coriolis force to be

$$z_c \sim 260\text{ m}$$