Refractive index of free electron gas (plasma) "Drude model"

1) calculate response of single electron

- no restoring force
- initially, no damping

$$m\ddot{x} = qE(t) = qE_0 e^{-i\omega t}$$

$$\to x(t) = \frac{q}{m}\frac{1}{(-i\omega)^2} E_0 e^{-i\omega t} + \underbrace{\cancel{c_1 t}}_{\text{initial vel}} + \underbrace{\cancel{c_2}}_{\text{initial } x}$$

$$= -\frac{q}{m\omega^2} E(t)$$

2) calculate polarization:

$$\vec{P} = N_e \vec{p} = -\frac{N_e e^2}{m\omega^2}\vec{E} = \chi_e \vec{E}$$

3) dielectric constant

$$\vec{D} = \epsilon \vec{E} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi\chi_e)\vec{E}$$

$$\epsilon = 1 - \frac{4\pi N_e e^2}{m\omega^2} = n^2$$

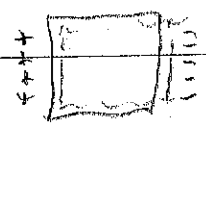
Define plasma frequency:

$$\omega_p^2 = \frac{4\pi N_e e^2}{m} = 4\pi N_e r_e c^2$$

Now

$$n(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \qquad \text{no collisions}$$

recall plasma frequency is natural charge separation oscillation freq.



these oscillations are a collective response
- not from an individual electron.

$\therefore$ can take previous result for bound electron with $\omega_0 \to 0$ (DC), $\beta \to 0$ no damping

Collisional effects:

electrons collide with ions
- in metal, electrons scatter from lattice defects or lattice vibrations (phonons)

collision rate $\gamma_c = 1/\tau$

Apply $\vec{E} = E_0 \hat{x}$ $\rightarrow$ accel in $\hat{x}$ direction. momentum $m\dot{x}$

collisions redirect this velocity randomly (avg)

$\rightarrow$ loss in momentum $\Delta p = -m\dot{x}$

$\rightarrow$ effective force $\frac{\Delta p}{\Delta t} = -\frac{m\dot{x}}{\tau} = -m\gamma_c\dot{x}$

$$\rightarrow m\ddot{x} = -eE(t) - \frac{m\dot{x}}{\tau}$$

$$\text{or} \quad \ddot{x} + \gamma_c \dot{x} = -\frac{eE(t)}{m}$$

γ_c plays role of damping β

with $x(t) = x_0 e^{-i\omega t}$

$$-\omega^2 x_0 - i\omega\gamma_c x_0 = -\frac{eE_0}{m}$$

$$x_0 = \frac{e/m}{\omega(\omega + i\gamma_c)} E_0$$

$$\vec{P} = \chi_e E = N_e(-e x_0) = \frac{-N_e e^2}{m\omega(\omega + i\nu_c)}$$

$$\varepsilon = 1 - \frac{4\pi N_e e^2}{m\omega(\omega + i\nu_c)} = 1 - \frac{\omega_p^2}{\omega(\omega + i\nu_c)}$$

Drift velocity: $\dot{x} = \frac{e(-i\omega)/m\, E_0}{\omega(\omega + i\nu_c)} = u_d$

→ current density:

$$J = N_e(-e)\,u_d = \frac{i N_e e^2/m}{\omega + i\nu_c} E$$

$$\vec{J} = \frac{N_e e^2/m}{1 - i\omega\tau} \vec{E} = \sigma \vec{E}$$

this defines the Drude conductivity.

at $\omega = 0$ (DC)

$$\sigma = \frac{N_e e^2}{m\tau}$$

mean free path (λ_{mfp})

cross section for collisions: $\sigma(v) \propto \frac{Ze^4}{m^2 v^4}$

what is velocity? $V = V_{th} + u_d$

- in a metal $V_{th} \sim V_{Fermi}$ (fast)

so $V_{th} \gg u_d$

$\lambda_{mfp} = 1/N_i\sigma$ (dimensionally $1/\lambda \sim \frac{1}{m^3} m^2$)

$$\nu_c = V/\lambda_{mfp} = N_i \sigma V_{th}$$

Wave propagation in plasmas:

- ignore collisions

$n = \sqrt{1 - \omega_p^2/\omega^2}$

for $\omega > \omega_p$ n is real, $n < 1$

$\omega >> \omega_p$

$n \approx 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}$

for $\omega < \omega_p$ n is imaginary $\omega << \omega_p$ $n \approx \frac{i\omega_p}{\omega}$

$E(z,t) \sim E_0 e^{i(k_0 n z - \omega t)}$

$\sim E_0 e^{i(\frac{\omega}{c} \cdot i \frac{\omega_p}{\omega} z - \omega t)} = E_0 e^{-\frac{\omega_p}{c} z} e^{-i\omega t}$

same form as evanescent wave

→ wave is reflected.

electrons move to shield E from penetrating.

examples:

ionosphere $N_e \approx 10^{11}\ m^{-3}$

$\omega_p \approx 1.8 \times 10^7$ rad/s

$\nu_p \approx 3$ MHz

AM waves ($\nu \sim 100$ kHz) are reflected.

metal $N_e \sim 10^{23}\ m^{-3}$

$\omega_p \sim 9 \times 10^{15}$ rad/s

$\lambda_p \sim 200$ nm

reflect visible light, transmit VUV