Advanced Engineering Mathematics		**Homework Three**
	Integration Review, Orthogonal Expansions, Introduction to Fourier Series	
Text: 11.1-11.2	Lecture Notes : 9	Lecture Slides: 4

Quote of Homework Four	
Willy Wonka: Oh, you can't get out backwards. You've gotta go forwards to go back. Better press on.	
	Roald Dahl : Willy Wonka and the Chocolate Factory (1971)

1. Integration Review

1.1. **Integration by Parts.** $\int x^3 \cos(5x)dx$

1.2. **Integration by ?** $\int x^2 \sin(2x^3)dx$

1.3. **Tricky IBP or Tricky Algebra.** $\int e^{ax} \cos(bx)dx$ and $\int e^{ax} \sin(bx)dx$

1.4. **Integration of *Delta 'Functions'*.** Justify that $\int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx = g(x_0)$ for some $x_0 \in \mathbb{R}$

1.5. **Integrals of *Gaussian Functions*.** Show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

1.6. **Orthogonality.** Show that $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0$ for all $n, m \in \mathbb{N}$

1.7. **More Orthogonality.** Show that $\int_a^b e^{i\frac{n\pi}{L}x} e^{-i\frac{m\pi}{L}x} dx = 2L\delta_{mn}$ where $L = \frac{b-a}{2}$ and for all $n, m \in \mathbb{Z}$.

2. Orthogonal Expansions

Given,

$$\hat{\mathbf{i}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \qquad \hat{\mathbf{j}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{1}$$

2.1. **Orthonormality.** Show that the vectors are orthonormal by verifying the inner-products $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1$.

2.2. **Orthogonal Representation I.** Show that any vector for $\mathbb{R}^2$ can be created as a linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$. That is, given,

$$\hat{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}, \tag{2}$$

show that c_1, c_2, can be found in terms of x_1 and x_2.

2.3. **Orthogonal Representation II.** Show that if $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx, \tag{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx, \tag{4}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{5}$$

3. Introduction to Fourier Series

3.1. **Wikipedia.** Go to `http://en.wikipedia.org/wiki/Fourier_series` and read the introductory material on Fourier Series and describe in your own words the purpose and application of Fourier Series.

3.2. **Graphing.** Using the Java Applet found at `http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Fourier/fourier.html`, use the applet to graph a truncated Fourier Series approximating the saw-tooth function. What occurs at the points jump-discontinuity?

3.3. **Truncated Fourier Series.** Read, as much as you can, of `http://en.wikipedia.org/wiki/Gibbs_phenomenon`. The sum of a finite, or infinite amount of periodic functions is periodic. Is this always true for both finite and infinite sums of continuous functions? Can you think of a counterexample? [1]

4. Fourier Series : Even

Let $f(x) = x^2$ for $x \in (-\pi, \pi)$ be such that $f(x + 2\pi) = f(x)$.

4.1. **Graphing.** Sketch f on $(-2\pi, 2\pi)$.

4.2. **Symmetry.** Is the function even, odd or neither?

4.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

4.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f.

5. Fourier Series : Oddish

Let $f(x) = x + \alpha$ for $x \in (-\pi, \pi)$ and $\alpha \in \mathbb{R}$ be such that $f(x + 2\pi) = f(x)$.

5.1. **Graphing.** Sketch f on $(-2\pi, 2\pi)$.

5.2. **Symmetry.** Is the function even, odd or neither?

5.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

5.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f assuming that $\alpha = 1$.

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[1] These questions are meant to lead you. Remembering that sine and cosine are examples of continuous periodic functions, you should be thinking about the following string of thoughts.

(1) Fourier series represent an 'arbitrary' periodic function in terms of known periodic functions.
(2) Increasing the number of terms in a Fourier series creates better and better sinusoidal wave-form fits of the function f and in the limit of infinitely many terms this fit is exact 'almost-everywhere'.
(3) Hopefully by the time you do this problem we would have mentioned in class that the Fourier series representation of a function converges in the sense of averages and that since jump-discontinuities are integrable-discontinuities the Fourier series would average the right and left hand limits of the function at the point of discontinuity. This will happen indifferent to the actual value of the function at the point of discontinuity. Thus the Fourier series may actually differ from its function at the boundaries of its periodic-domains! In this way we take $=$ to mean equality *almost everywhere* (`http://en.wikipedia.org/wiki/Almost_everywhere`).

So, we have that the sawtooth example from class and the square-wave example online are examples where the infinite sum of continuous periodic functions converges to a periodic function with jump-discontinuities.