

Homework #1 Solution:

1.
Given

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a)

$$\sigma_1^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_2^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_3^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b) Note:

- $[B, A] = BA - AB = -(AB - BA) = -[A, B]$
 - $[\sigma_i, \sigma_i] = \sigma_i^2 - \sigma_i^2 = 0$
- Thus we only need to consider, $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$

$$[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 =$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\sigma_3$$

and

$$2i \sum_{k=1}^3 \epsilon_{12k} \sigma_k = 2i\epsilon_{123} \sigma_3 = 2i\sigma_3$$

$$[\sigma_1, \sigma_3] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -2i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -2i\sigma_2$$

and

$$2i \sum_{k=1}^3 \epsilon_{13k} \sigma_k = 2i(\epsilon_{131} \sigma_1 + \epsilon_{132} \sigma_2 + \epsilon_{133} \sigma_3) = -2i\sigma_2$$

$$[\sigma_2, \sigma_3] = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = 2i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2i\sigma_1$$

and

$$\sum_{k=1}^3 \epsilon_{23k} \sigma_k = 2i \epsilon_{131} \sigma_1 = 2i \sigma_1$$

$$2i \sigma_3 = [\sigma_1, \sigma_2] = -[\sigma_2, \sigma_1] = -2i \sum_{k=1}^3 \epsilon_{21k} \sigma_k =$$

$$= -2i \epsilon_{213} \sigma_3 = 2i \sigma_3$$

c) Similarly

$$\{\sigma_i, \sigma_i\} = \sigma_i^2 + \sigma_i^2 = 2\sigma_i^2 = 2I = 2I \delta_{ii}, \quad i = 1, 2, \dots$$

and

$$\{\sigma_i, \sigma_j\} = \{\sigma_j, \sigma_i\}$$

and

$$\{\sigma_1, \sigma_2\} = \sigma_1 \sigma_2 + \sigma_2 \sigma_1 = 0$$

$$\{\sigma_1, \sigma_3\} = \sigma_1 \sigma_3 + \sigma_3 \sigma_1 = 0$$

$$\{\sigma_2, \sigma_3\} = \sigma_2 \sigma_3 + \sigma_3 \sigma_2 = 0$$

2. We begin by writing the 3 linear equations (3),(4),(5) as the augmented matrix,

$$\left[\begin{array}{ccc|c} 6 & 18 & 1 & 20 \\ -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \end{array} \right] \begin{array}{l} R1 \rightarrow R3 \\ R3 \rightarrow R2 \\ R2 \rightarrow R1 \end{array} \sim \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 5 & 15 & -9 & 11 \\ 6 & 18 & -4 & 20 \end{array} \right] \sim$$

$$\sim \begin{array}{l} R2=5R1+R2 \\ R3=6R1+R3 \end{array} \left[\begin{array}{ccc|c} -1 & -3 & 8 & 4 \\ 0 & 0 & 40-9 & 11+20 \\ 0 & 0 & 44 & 44 \end{array} \right] \sim \begin{array}{l} R3=R3/44 \\ R2=R2/44 \end{array}$$

$$\sim \begin{array}{l} R1=-R1 \\ R3=R3+R2 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -8 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 8 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim$$

$$\sim \begin{array}{l} R1=R1+8R2 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Which corresponds to the row equivalent linear system,

$$x_1 + 3x_2 = 4$$

$$x_3 = 1$$

Letting $x_2 = t$ implies that the general solution set is given by,

$$(\star) = \begin{array}{l} x_1 = -3t + 4 \\ x_2 = t \\ x_3 = 1 \end{array}, \quad t \in \mathfrak{R}$$

With x_1 dependent on the one free variable x_2
 (★) parameterizes a 2-D line in 3-D space

3.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array} \right] \sim_{R3=R3-3R1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array} \right]$$

corresponds to the linear system

$$\begin{aligned} x_1 + 3x_2 &= 2 \\ (h-9)x_2 &= k-6 \end{aligned}$$

a) For this system to be consistent with a unique solution,

$$(h-9)x_2 = k-6 \Rightarrow x_2 = \frac{k-6}{h-9}, \text{ assuming } h-9 \neq 0$$

thus $h-9 \neq 0 \Rightarrow h \neq 9$ will yield no free variables and the linear system is consistent with a unique point of intersection of the two lines.

b) For infinitely many solutions (for x_2 to be a free variable) we require that

$$(h-9)x_2 = k-6 \Leftrightarrow 0 \cdot x_2 = 0 \Rightarrow h=9, k=6$$

Thus x_2 is free.

c. For no solutions we require,

$$(h-9)x_2 = k-6 \Leftrightarrow 0 \cdot x_2 = c, c \in \mathfrak{R}, c \neq 0$$

This implies that $h=9$ and $k \neq 6$. Thus the augmented column is a pivot column and the system has no solutions.

4. We have the following augmented matrix representation of (1) and (2),

$$\begin{aligned} & \left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array} \right] \sim_{\substack{R1=dR1-bR2 \\ R1=aR2-cR1}} \left[\begin{array}{cc|c} ad-cb & bd-db & df-bg \\ ac-ca & ad-cb & dg-cf \end{array} \right] = \\ \stackrel{(\star)}{=} & \left[\begin{array}{cc|c} ad-cb & 0 & df-bg \\ 0 & ad-cb & ag-cf \end{array} \right] \sim_{\substack{R2=R2/(ad-cb) \\ R1=R1/(ad-cb)}} \left[\begin{array}{cc|c} 1 & 0 & \frac{df-bg}{ad-cb} \\ 0 & 1 & \frac{ag-cf}{ad-cb} \end{array} \right] \end{aligned}$$

which is equivalent to the linear system,

$$\begin{aligned} (1') \quad & 1 \cdot x_1 + 0 \cdot x_2 = x_1 = \frac{df-bg}{ad-cb} \\ (2') \quad & 0 \cdot x_1 + 1 \cdot x_2 = \frac{ag-cf}{ad-cb} \end{aligned}$$

Note, to do the division at (★) we have assumed that

$$ad-bc \neq 0$$

This is a common statement which places restriction on a,b,c,d.

5. If $\vec{b} = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}$ is a linear combination of the vectors, $\vec{a}_1 = \begin{bmatrix} 5 \\ -4 \\ 9 \end{bmatrix}$,
 $\vec{a}_2 = \begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix}$ formed from the columns of $A = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}$ then there must
 exist $x_1, x_2 \in \mathfrak{R}$ such that

$$\vec{a}_1 \cdot x_1 + \vec{a}_2 \cdot x_2 = \vec{b} \Leftrightarrow \begin{cases} 5x_1 + 3x_2 = 22 \\ -4x_1 + 7x_2 = 20 \\ 9x_1 - 2x_2 = 15 \end{cases}$$

To determine if this is true we row reduce the augmented matrix,

$$\begin{aligned} \left[\begin{array}{cc|c} 5 & 3 & 22 \\ -4 & 7 & 20 \\ 9 & -2 & 15 \end{array} \right] &\xrightarrow{\substack{R3=5R3-9R1 \\ R2=5R2+4R1}} \left[\begin{array}{cc|c} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & -37 & -123 \end{array} \right] \sim \\ &\xrightarrow{R3=47R3+37R2} \left[\begin{array}{cc|c} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & 0 & 1175 \end{array} \right] \end{aligned}$$

which corresponds to the linear system,

$$\begin{cases} 5x_1 + 3x_2 = 22 \\ 47x_2 = 188 \\ 0 \cdot x_2 = 1175 \end{cases} \quad (\star)$$

There is no x_2 such that (\star) can be satisfied. Thus, the linear system is inconsistent and \vec{b} is not a linear combination of \vec{a}_1 and \vec{a}_2 .