Second Order Linear Equations

Quote of Second Order Linear Equations Homework	
When I get to the bottom I go back to the top of the slide. Where I stop and turn and I go for a ride 'till I get to the bottom and I see you again.	
	The Beatles : The White Album (1968)

0. Introduction

The following homework concentrates on the second order linear equation,

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t), \tag{0.1}$$

where a, b, c are smooth functions of time and f is a piecewise smooth or distributional in nature.[1] From class we understand that the general solution to such an equation is given by

$$y(t) = y_h(t) + y_p(t), \tag{0.2}$$

where $y_h(t)$ is the general solution to the corresponding homogeneous problem of (0.1) and y_p is exactly one particular solution to (0.1). Moreover, we know that (0.1), where $f(t) \equiv 0$ defines a two-dimensional solution space, which implies that we search for homogeneous solutions of the form,

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t), \quad c_1, c_2 \in \mathbb{R}, \tag{0.3}$$

where y_1, y_2 are both solutions to the homogeneous ODE. Also, we understand that if you know one homogeneous solution to (0.1) then a second linearly independent solution can be found via

$$y_2(t) = k(t)y_1(t), \ k(t) = \int \frac{p(t)}{[y_1(t)]^2}dt, \ p(t) = e^{-\int (b(t)/a(t))dt}, \tag{0.4}$$

which gives the general homogeneous solution. Once this is known we can then find the particular solution via

$$y_p(t) = y_2 \int \frac{f(t)y_1(t)}{a(t)W(t)}dt - y_1 \int \frac{f(t)y_2(t)}{atx)W(t)}dt, \ W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t), \tag{0.5}$$

That being said, I need to emphasize that this is *the hard way* and that these formulae should only be used as a last resort. I say this because it is always possible to verify proposed solutions to differential equations by direct substitution. This means that if there exists a regimented system of guessing then we can exploit this to find solutions without resorting to (0.4) and (0.5). Just as before, we motivate (0.1) with some mathematical models stemming from Newton's second law.[2]

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[1] We won't need the idea of a *distribution* quite yet and when we get there we won't need any theory. So, just let that word float around for a while.

[2] A body of mass m subject to a net force F undergoes an acceleration a that has the same direction as the force and a magnitude that is directly proportional to the force and inversely proportional to the mass, i.e., $F = ma$. Alternatively, the total force applied on a body is equal to the time derivative of linear momentum of the body.

Quote of Linear v. Nonlinear	
Professor Hubert Farnsworth: Good news, everyone. You'll be making a delivery to the planet Trisol. A mysterious planet located in the mysterious depths of the Forbidden Zone. **Leela**: Professor, are we even allowed in the Forbidden Zone? **Professor Hubert Farnsworth**: Why of course. It's just a name, like the Death Zone, or the Zone of No Return. All the zones have names like that in the Galaxy of Terror.	
	Futurama: My Three Suns (1999)

1. Nonlinear Oscillations and Connections to Linear Oscillations

In class we presented a derivation of the linear mass-spring equation. The critical assumptions were

(1) The force due to friction is proportional to the velocity. [3]
(2) The force due to the spring is proportional to the displacement of the mass from the mass-spring equilibrium state.[4]

For now we won't abandon these assumptions but we will try to achieve nonlinearity another way.

1.1. **Oscillating Pendulum.** Consider a mass, m, attached to one end of a rigid, but weightless, rod of length L. The other end of the rod is supported at the frictionless origin O, and the rod is free to rotate in the plane. The position of the pendulum is described by the angle, θ, between the rod and the downward vertical direction, with the counterclockwise direction taken as positive.

1.1.1. *Pendulum Equation.* Using the previous description, draw a diagram of the system and by appealing to Newton's second law show that the equation of motion associated with the motion on θ is given by

$$(1.1) \qquad \frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\sin(\theta) = f(t),$$

where the coefficient of kinetic friction is given by $c = mL\gamma$, $\omega^2 = g/L$ and f is an arbitrary external force. Classify this differential equation by type, order and linearity.

1.1.2. *Small Angle Approximation.* There is no way to solve (1.1) generally, which is typical of higher order nonlinear equations. However, we can apply to so-called *small angle approximation*. To do this rewrite (1.1) replacing $\sin(\theta)$ with its Taylor series. If θ is less than one then the higher order terms decrease rapidly and can be discarded. The highest power kept is called the *order* of the approximation. Write down the ODE associated with first-order approximation to the sine function. What equation is this? Can it be solved?

1.1.3. *Nonlinear Effects: Conservative Case.* So, what can be done when we keep either some or all of the nonlinear terms? Well, not a lot generally. However, if we turn off the viscosity/dampening and any external forces then we expect that any energy we put into the pendulum stays in the pendulum for all time. That is, we expect that the system is conservative. Write down the undamped and unforced version of (1.1) and, using the energy method described in class, derive the conservation law associated with the ODE.

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[3] As we saw earlier in the semester, this assumption can be questioned and in more sophisticated fluid models we appeal to Lord Rayleigh's drag equation, which is quadratic in the velocity variable. See: `http://en.wikipedia.org/wiki/Drag_equation`

[4] Just as with friction, the assumption can be replaced in favor of a more precise model. Doing so essentially asks whether the spring is in the elastic regime, `http://en.wikipedia.org/wiki/Elastic_limit`, which is characterized by Hooke's law, `http://en.wikipedia.org/wiki/Hooke's_law`, where strain is directly proportional to stress. Outside of this limit or for springs in more exotic situations, Hook's law is not a good approximation and nonlinear effects must be taken into account.

1.2. **Undamped, Unforced Pendulum: Solutions.** It is possible to tease a bit of information out of the undamped, unforced pendulum equation,

$$\ddot{\theta} + \omega^2 \sin(\theta) = 0. \tag{1.2}$$

The starting point is its associated conservation law,

$$\frac{\dot{\theta}^2}{2} - \frac{\omega^2}{2}\cos(\theta) = E \in \mathbb{R}, \tag{1.3}$$

which is a first-order separable ODE.

1.2.1. *Separation of Variables I: Elliptic Integrals.* Using separation of variables find $\theta(t)$.[5]

1.2.2. *Separation of Variables II: Small Angle Approximation.* Assume $\theta \ll 1$ and, using the quadratic approximation to cosine, find $\theta_2(t)$, which solves the second order approximation to the ODE.

1.2.3. *Separation of Variables III: Nonlinear Effects, Fail.* Assume $\theta^2 \ll 1$ and, using the quartic approximation to cosine, find $\theta_4(t)$, which solves the fourth order approximation to the ODE.[6]

Quote of LRC Circuits	
Homer Simpson: A big mountain of sugar is too much for one man. I can see now why God portions it out in those little packets.	
	The Simpsons: Lisa's Rival, S06E02 (1994)

2. Oscillations and LRC Circuits

For this problem we develop a correspondence between our mass-spring oscillations and circuits that have inductors, resistors and capacitors in series. Such a circuit is called and LRC circuit and will behave very much like a mass-spring system. For more information see:

- General Information: `http://en.wikipedia.org/wiki/RLC_circuit`
- Circuit Diagram: `http://en.wikipedia.org/wiki/File:RLC_series_circuit.png`
- LRC in Series: `http://en.wikipedia.org/wiki/RLC_circuit#Series_RLC_circuit`

2.1. **LRC Model.** In reference to the circuit diagram we mention that the current I, measured in amperes, is a function of time t. The resistance R (ohms), the capacitance C (farads), and the inductance L (henrys) are all positive and assumed to be known. The impressed voltage V (volts) is a given function of time. We will also need to understand the total charge Q (coulombs) on the capacitor at time t. The relation between charge Q and current I is $I = Q'$.

To arrive at the model equation we apply Kirchhoff's second law, which states that in a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit. Another way of saying this is that the sum of potential voltage differences across a closed circuit must be zero. [7] Now, we need only understand the voltage drops across the components. To this end we have:

- A resistor will slow the current according to: $I = v_R/R$
- A capacitor will store up charge and create a voltage difference according to: $v_C = Q/C$
- An inductor will create a voltage drop proportional to the time rate of change of the current according to: $v_L = LI'$

2.1.1. *Model Equation.* Using Kirchhoff's second law and the known information relating voltage drops to the circuit elements, derive the model equation for current I in the circuit as a function of time.

2.1.2. *Undriven LRC.* Assume that $V(t) = 0$ for all time and solve the differential equation $LI'' + RI' + C^{-1}I = V(t)$ for $I(t)$ and show that in the presence of resistance the current is a decaying function of time.

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[5] This case has been studied and can be managed through the so-called elliptic integrals. See: `http://en.wikipedia.org/wiki/Elliptic_integral`

[6] Actually, you will just want to write down the integral. I can't solve it by hand. :(In fact, I tried a CAS system and it didn't much like it either. The moral here is that nonlinear terms are bad. Update! I'm still working on it and I think there is hope. Extra credit for those who want to journal some ideas about it. Ideas concluding in a solution will be pinned to my fridge.

[7] This is similar to the idea that the line integral for a loop in a conservative field must be zero.

2.1.3. *Driven Vibrations.* Assume that $V(t) = F_0 \cos(\omega_0 t)$, $\omega_0 \in \mathbb{R}$ and solve for $I(t)$.

2.1.4. *Phase Angle.* The homogeneous solution is often called the transient solution since it decays in time. The particular solution is often called the steady-state solution since it persists. It is common to report the stead-state solution as one shifted trigonometric function containing a so-called phase angle. Specifically, show that $I_p(t)$ can be written as $I_p(t) = A\cos(\omega_0 t - \delta)$ where

$$(2.1) \qquad A = \frac{F_0}{\Delta}$$

$$(2.2) \qquad \cos(\delta) = \frac{L(\omega^2 - \omega_0^2)}{\Delta}$$

$$(2.3) \qquad \sin(\delta) = \frac{R\omega}{\Delta}$$
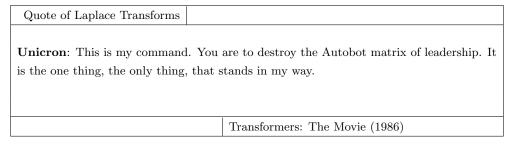
$$(2.4) \qquad \Delta = \sqrt{L^2(\omega^2 - \omega_0^2)^2 + R^2 \omega_0^2}$$

$$(2.5) \qquad \omega^2 = \frac{1}{LC}$$

This allows one to understand the steady-state response in terms of an amplitude A at the cost of a cosine shifted be δ, which is called the phase angle. [8]

2.1.5. *Steady-State Analysis.* What is A for $\omega_0 \to 0$? What is A for $\omega_0 \to \infty$?[9]

2.1.6. *Maximum Steady-State Amplitude.* Find the value of ω_0 for which the amplitude A is maximum. Find A at this value. What occurs to this amplitude when $R \to 0$?

Quote of Laplace Transforms	
Unicron: This is my command. You are to destroy the Autobot matrix of leadership. It is the one thing, the only thing, that stands in my way.	
	Transformers: The Movie (1986)

3. Laplace Transforms

Quote of Laplace Transforms	
Morpheus:You're here because you know something. What you know you can't explain, but you feel it. You've felt it your entire life, that there's something wrong with the world. You don't know what it is, but it's there, like a splinter in your mind, driving you mad.	
	The Matrix: (1999)

4. Phase Analysis of Systems and the Trace Determinant Plane

The most general setting for the study linear ordinary differential equations is the construct of linear algebra. The key idea is that an arbitrary n^{th}−order linear ODE can be mapped onto a system of n−many linear first order equations. It is through these ideas that higher dimensional phase analysis takes shape but before we get into this we should start with some simple definitions. First we introduce the notion of a linear system of equations.

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[8]The idea is that we know the response is going to be trigonometric in nature and the δ is just an offset on where to begin the oscillations. The point is that we are most concerned about amplitude.

[9]This will show that for low frequency forcing the amplitude has a nonzero finite value and that for high frequency forcing the amplitude tends to zero. This now begs the question, is there a frequency between such that the amplitude achieves a maximum?

Definition 1. $m-$many linear equations of $n-$many variables

The following equations

(4.1)
$$\begin{aligned}
a_{11}x_1+ & \quad a_{12}x_2+ & \quad a_{13}x_3+ & \quad \cdots & \quad +a_{1n}x_n & \quad = b_1, \\
a_{21}x_1+ & \quad a_{22}x_2+ & \quad a_{23}x_3+ & \quad \cdots & \quad +a_{2n}x_n & \quad = b_2, \\
a_{31}x_1+ & \quad a_{32}x_2+ & \quad a_{33}x_3+ & \quad \cdots & \quad +a_{3n}x_n & \quad = b_3, \\
\vdots & \quad \vdots & \quad & \quad \ddots & \quad \vdots & \quad \vdots \\
a_{m1}x_1+ & \quad a_{m2}x_2+ & \quad a_{m3}x_3+ & \quad \cdots & \quad +a_{mn}x_n & \quad = b_m,
\end{aligned}$$

form a system of $m-$many *linear* equations of $n-$many variables. Each equation is a linear combination of the x_i variables, the a_i's are the weights/coefficients of the linear combination and the b_i's are the inhomogeneous terms.

Remark 1. Typically, $a_i, b_i \in \mathbb{R}$ but they really could be any well-defined mathematical object. This is important for general ODE, but for ease of use just think about them as constants.

Now, equation (4.1) is important because it highlights the linear nature of the equations,[10] but it is not very compact and consequently difficult to algebraically manipulate. So, we seek a more compact notation and with that in mind we define the following mathematical object.

Definition 2. An m by n matrix

We say $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an m by n matrix and write

(4.2)
$$\mathbf{A} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & & & \ddots & \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}$$

with entries $a_{ij} = [\mathbf{A}]_{ij} \in \mathbb{R}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

Remark 2. A matrix is nothing more than standard mathematical notation for a table/array of data. This data can be real numbers, complex numbers, functions, whatever. The only requirement is that the entries be well defined mathematical objects. '

Remark 3. It is useful to notice that vectors and scalars are special cases of a matrix where $n = 1$ and $m = n = 1$, respectively.

To see the correspondence between matrices and linear systems we must first define an new algebraic operation.[11]

Definition 3. Matrix Multiplication

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{p \times q}$. The product $\mathbf{AB}$ is defined if and only if $p = n$ and the resulting matrix $\mathbf{AB} = \mathbf{C} \in \mathbb{R}^{m \times q}$ has elements defined by

(4.3)
$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

for $i = 1, 2, 3, \ldots m$ and $j = 1, 2, 3, \ldots q$.

Remark 4. So, we must remember that a matrix product is defined only when the number of columns in the matrix on the left is equal to the number of rows in the matrix on the left.

Remark 5. It is important to note that the $i, j-$element of $\mathbf{C}$ is defined to be the dot-product of the $i^{th}-$row of $\mathbf{A}$ and the $j^{th}-$column of $\mathbf{B}$. This is why the dimensions of the matrix must obey the given constraint.

Remark 6. An immediate consequence of this definition is that matrix multiplication is not generally commutative. That is $\mathbf{AB}$ may not be the same as $\mathbf{BA}$ for some matrices. To see this, quickly, let $\mathbf{A} \in \mathbb{R}^{2 \times 3}$ and let $\mathbf{B} \in \mathbb{R}^{3 \times 2}$. With these definitions we have that $\mathbf{AB} \in \mathbb{R}^{2 \times 2}$ while $\mathbf{BA} \in \mathbb{R}^{3 \times 3}$, which shows that they clearly cannot be equal![12]

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[10] To see this take a derivative of each equation with respect to the x_i variable and you will see that this is a constant. Any function with a constant rate of change is linear.

[11] Actually, this operation won't be new. It is just a lot of old operations.

[12] It can get worse, consider the same example where $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ now $\mathbf{BA}$ isn't even defined.

Implication 1. *Now, we have our result. The linear system, equation (4.1), is equivalent to the matrix equation,*

$$
(4.4) \qquad \boldsymbol{A}\boldsymbol{x} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & & & \ddots & \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m
\end{bmatrix}
= \boldsymbol{b},
$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{x} \in \mathbb{R}^{n \times 1} = \boldsymbol{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$, which is the most compact notation associated with equation (4.1). Using this representation the linear system can now be seen as the algebraic problem, given $\boldsymbol{A}$ and $\boldsymbol{b}$ can you find $\boldsymbol{x}$? [13]

So, what now? Well, let's see how this interfaces with our study of ODE.

4.1. **Phase Space Analysis of Mass-Spring Systems.** Recall the mass-spring equation,

$$
(4.5) \qquad my'' + \gamma y' + ky = f(t), \quad m, \gamma, k \in \mathbb{R}^+,
$$

which is a second order linear ODE with constant coefficients. The following problems will recast this equation into a 2×2 linear system of coupled first order problems and from this perspective define a phase plane, which can be used to analyze the flow defined by the equation. We begin with some statements about (4.5) before we move into the plane.

4.1.1. *Equilibrium Solutions.* Show that if there is a spring in the physical problem then the homogeneous equation has only one equilibrium solution. What is this solution and what does it mean physically?

4.1.2. *Conversion to a First-Order System.* Define the velocity variable $v = y'$ and using it find constants $a, b, c, d \in \mathbb{R}$ and functions f_1, f_2 such that

$$
(4.6) \qquad \frac{d\mathbf{Y}}{dt} = \frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \mathbf{AY} + \mathbf{F},
$$

is equivalent to equation (4.5).

4.1.3. *Homogeneous Problem - Decomposition.* We now study the associated autonomous case

$$
(4.7) \qquad \mathbf{Y}' = \mathbf{AY}.
$$

Assume an exponential solution of the form $\mathbf{Y}(t) = \mathbf{x}e^{\lambda t}$ where $\mathbf{x} \in \mathbb{R}^2$ and show that equation (4.7) is equivalent to

$$
(4.8) \qquad \lambda\mathbf{x} = \mathbf{Ax},
$$

which defines an eigenvalue, λ, eigenvector, $\mathbf{x}$, problem.

4.1.4. *Homogeneous Problem - Recasting.* The idea is that if we can find λ and $\mathbf{x}$ then we know our solution. Clearly, $\mathbf{x} = \mathbf{0}$ solves equation (4.8), but we want a dynamic solution and therefore we seek to disregard this trivial case. To find this we show that (4.8) can be recast into

$$
(4.9) \qquad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}
$$

where $\mathbf{I}$ is the so-called two-by-two identity matrix given by,

$$
(4.10) \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Specifically, start with (4.9) and show that $\mathbf{Ix} = \mathbf{x}$ and rearrange to get (4.8).

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[13]We won't ask this, per se, but it is useful to note that the definition of matrix and matrix multiplication has maximally compactified the linear system. Also, if we think about (4.1) as $m-$many $(n-1)-$many hyperplanes then finding $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$ is equivalent to asking whether the hyperplanes simultaneously intersect one another in $n-$dimensional space. Anyhow, if this interests you then you should take MATH332-Linear Algebra.

4.1.5. *Homogeneous Problem - Eigenvalues: Characteristic Polynomial.* Again, it should be clear the $\mathbf{x} = \mathbf{0}$ is a solution to (4.8). Disregarding this trivial solution we must impose some sort of condition so that there are nontrivial solutions. Linear algebra tells us that for this to occur the matrix $\mathbf{A} - \lambda\mathbf{I}$ must have a determinant that vanishes.[14] That is, to find nontrivial eigenvectors we require

$$(4.11) \qquad\qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

which is called the characteristic equation associated with the ODE. Using the coefficient matrix defined by your mass-spring problem, find the associated characteristic equation and, from this equation, find the eigenvalues for the problem.

4.1.6. *Homogeneous Problem - Eigenvectors.* Now that you have the eigenvalues it is possible to find the vectors $\mathbf{x}$ that satisfies (4.9). Find these eigenvectors.[15]

4.1.7. *Homogeneous Problem - Classification.* Getting back to phase space, the solution $\mathbf{Y} = \mathbf{x}e^{\lambda t}$ defines a trajectory in this space. Moreover, the eigenvector defines a direction in this space, while the eigenvalue defines the time dynamics on this line. If we are only concerned about the time dynamics then, really, the eigenvector is in some sense not needed. With that in mind. Classify the fixed point(s) of the system based on the physical parameters m, γ, k. Specifically, show that for a mass-spring system the equilibrium structure is always either a sink, center or nodal line.

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[14]Recall that we showed during our discussion with the Wronskian that a vanishing determinant implies that the vectors, in 2-space, are colinear. If you think about vectors in 2-space then it should be clear that they all have a common point of intersection, which is the origin, and if we want to get more than just this point we must make them colinear. This is the geometric point of the determinant.

[15]It turns out that for two-by-two do not need any advanced techniques. Since $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ the vectors represented by the rows of the matrix must be colinear. Thus, to get this direction (x_1, x_2) you need only write out the linear equation for a single row and then pick a value for one variable. After this the second variable is known from the equation. The relationship between the two variables defines a rise and a run (AKA a slope), which, since a vector must start at the origin, is everything we need to define the eigenvector. It should be noted that this trick will not work in 3-space and higher.