Classical Vector Spaces

Linear Spaces Defined by A $\in \mathbb{R}^{m\times n}$

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Overview/Keywords/References

Advanced Engineering Mathematics **Slide Set Two**

Linear Vector Spaces : Basis, Dimension, Null, Column and Row-Spaces

Reference Text: EK 7.4, 7.9 **Example: Rank-Nullity Theorem**

- See Also:
  - Lecture Notes : 05.LN.Introduction to Linear Vector Spaces
- Start:
  - Homework Two

Before We Begin

Quote of Slide Set Two
The Dude: Yeah, my thinking about the case, man, it had become uptight. Yeah.
The Big Lebowski : Cohen Brothers (1998)

Assumptions - Part Ie

At this point it is assumed that you have studied the system of linear equations, $\mathbf{Ax} = \mathbf{b}$, by finding its pivot structure through row-reduction. Specifically, one should have reached the following conclusions:

- If $\mathbf{A} \in \mathbb{R}^{m \times n}$ then $\mathbf{x} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m$ and the following statements are equivalent:
  - $\mathbf{A}$ has a pivot in every row
  - $\mathbf{Ax} = \mathbf{b}$ is solvable for every $\mathbf{b} \in \mathbb{R}^m$
  - Every $\mathbf{b} \in \mathbb{R}^m$ can be written as a linear combination of the columns of $\mathbf{A}$
  - The set of m-many hyper-planes simultaneously intersect for at least one point in $\mathbf{R}^n$

Assumptions - Part II

- Underdetermined Systems : If $m < n$ for $\mathbf{A} \in \mathbb{R}^{m \times n}$ then we say the system is underdetermined.
  - If an underdetermined system has a solution then this solution cannot be unique.
- Overdetermined Systems : If $m > n$ for $\mathbf{A} \in \mathbb{R}^{m \times n}$ then we say the system is overdetermined.
  - An overdetermined system cannot have a solution for every $\mathbf{b} \in \mathbb{R}^m$.
- Square Systems : If $m = n$ for $\mathbf{A} \in \mathbb{R}^{m \times n}$ then we say the system is square.
  - It is possible for a square system to have a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.

Assumptions - Part III

Square systems are particularly interesting since there is the possibility of existence <u>and</u> uniqueness for every $\mathbf{b} \in \mathbb{R}^n$. The following statements are equivalent and, when taken together, is typically called the invertible matrix theorem (IMT):

- $\mathbf{A}$ is an invertible matrix
- $\mathbf{A}^{-1}$ exists
- $\mathbf{A} \sim \mathbf{I}$
- A solution to $\mathbf{Ax} = \mathbf{b}$ exists and this solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
- The hyper-planes intersect at only one point in $\mathbb{R}^n$
- Every $\mathbf{b} \in \mathbb{R}^n$ can be written as a linear combination of the columns from $\mathbf{A}$
- $\det(\mathbf{A}) \neq 0$

Definitions - Part I

Since not every system will have square coefficient data a more general construct is required to discuss solubility of linear systems. This construct will use the following definitions:

- Linear Combination : We say that $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of vectors from $S = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n\}$ if there exist scalars $x_1, x_2, x_3, \ldots, x_n$ such that,

$$\mathbf{b} = \sum_{i=1}^{n} x_i \mathbf{a}_i = x_i \mathbf{a}_i = \mathbf{A}\mathbf{x} \tag{1}$$

  - The set of all such **b** defines the spanning set of S

$$span(S) = \left\{ \mathbf{b} \in \mathbb{R}^m : \mathbf{b} = \sum_{i=1}^{n} x_i \mathbf{a}_i \right\} \tag{2}$$

Definitions - Part II

Of the vectors in S it is possible that not all of them are needed to represent every element in $span(S)$. In order to determine those vectors from S needed to represent every element in $span(S)$ we define:

- Linear Independence : We say that a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots \mathbf{v}_k\}$ forms a linearly independent set if and only if

$$\mathbf{0} = \sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{V}\mathbf{c}, \tag{3}$$

has only the trivial solution $\mathbf{c} = \mathbf{0}$.

Definitions - Part III

Spanning sets are an example of a so-called ‘vector-space’ and since every nontrivial spanning set contains an infinite amount amount of vectors we must define some way to characterize different spanning sets. To do this we define the following:

- Basis : Given a spanning space we define a basis for this space to be any collection of linearly independent vectors, from the space, that spans the space.
- Dimension : Given a spanning space we define the dimension of this space to be the number of vectors in any basis for this space.

Key Point: Any two vector spaces, which have the same dimension are structurally identical.

Definitions - Part IV

The previous terminology now allows us to define the following spaces, which are important to the study of $\mathbf{A}\mathbf{x} = \mathbf{b}$.

- Null-Space : Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ we define the null-space of $\mathbf{A}$, $\text{Nul}(\mathbf{A})$, to be the set of all solutions to $\mathbf{A}\mathbf{x} = \mathbf{0}$.
  - The null-space is a vector space.
  - The null-space describes how the linear objects simultaneously intersect.
  - A basis for the null-space is found by explicitly solving the homogeneous equation.
  - The dimension of the null-space is the number of free-variables in the system.

Definitions - Part V

- Column-Space : Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ we define the column-space of $\mathbf{A}$, Col$(\mathbf{A})$, to be the space of all linear combinations of the columns of $\mathbf{A}$.
  - The column-space is a vector space.
  - The column-space describes the set of all inhomogeneous vectors for which $\mathbf{Ax} = \mathbf{b}$ is solvable.
  - A basis for the column-space is the set of linearly independent columns from $\mathbf{A}$.
  - The dimension of the column-space is the number of linearly independent columns of $\mathbf{A}$.
  - Since row-reduction does not change the solution to a linear system it does not change dependency relations. That is, if $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution and $\mathbf{A} \sim \mathbf{B}$ then $\mathbf{Bx} = \mathbf{0}$ has only the trivial solution.

Definitions - Part VI

- Row-Space : Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ we define the row-space of $\mathbf{A}$, Row$(\mathbf{A})$, to be the space of all linear combinations of the rows of $\mathbf{A}$.
  - A basis for the row-space is the set of linearly independent rows from $\mathbf{A}$.
  - The dimension of the row-space is the number of linearly independent rows of $\mathbf{A}$.
  - Row-reduction is reversible. That is, if $\mathbf{A} \sim \mathbf{B}$ then the linearly independent rows of $\mathbf{B}$ can be used as a basis for Row$(\mathbf{A})$.

General Framework for $\mathbf{Ax} = \mathbf{b}$

Using the idea of vector spaces we now have the following statements, which completely characterizes the general linear problem $\mathbf{A}_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$:

- $\mathbf{Ax} = \mathbf{b}$ is solvable if and only if $\mathbf{b} \in$Col$(\mathbf{A})$
- If $\mathbf{Ax} = \mathbf{b}$ is solvable and the dimension of Nul$(\mathbf{A})$ is zero then $\mathbf{Ax} = \mathbf{b}$ is uniquely solvable.
- Rank-Nullity Theorem: Rank $\mathbf{A}$ + dim$($Nul$(\mathbf{A})) = n$ where Rank $\mathbf{A}$=dim$($Col$(\mathbf{A}))$=dim$($Row$(\mathbf{A}))$.

Invertible Matrix Theorem : Redux

With this framework we can make the following additions to the IMT:

- $\mathbf{A}_{n\times n}$ is an invertible matrix
- $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n\} = \mathbb{R}^n$
- A basis for the column-space of **A** is a basis for $\mathbb{R}^n$
- Rank **A** $= n$
- The dimension of the null-space of **A** is zero
- $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution