

## **Classical Vector Spaces**

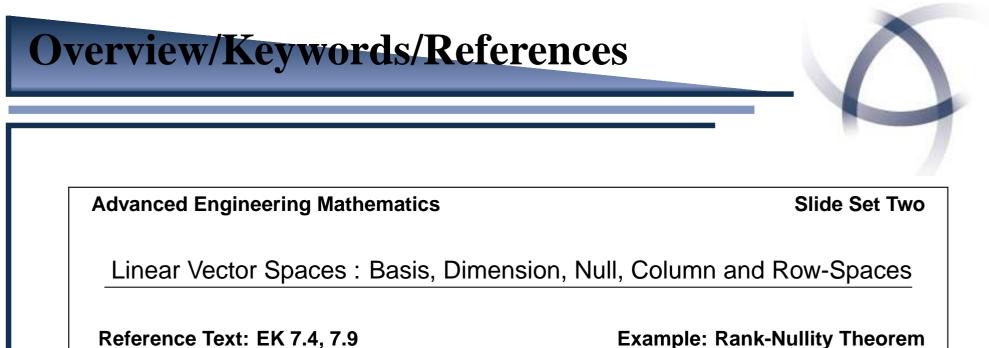
Linear Spaces Defined by  $A \in \mathbb{R}^{m \times n}$ February 2, 2010

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**Example: Rank-Nullity Theorem** 

- See Also:
  - Lecture Notes : 05.LN.Introduction to Linear Vector Spaces
- Start:
  - Homework Two

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Quote of Slide Set Two

The Dude: Yeah, my thinking about the case, man, it had become uptight. Yeah.

The Big Lebowski : Cohen Brothers (1998)

At this point it is assumed that you have studied the system of linear equations, Ax = b, by finding its pivot structure through row-reduction. Specifically, one should have reached the following conclusions:

- If  $\mathbf{A} \in \mathbb{R}^{m \times n}$  then  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{b} \in \mathbb{R}^m$  and the following statements are equivalent:
  - A has a pivot in every row
  - $\cdot$   $\mathbf{A}\mathbf{x} = \mathbf{b}$  is solvable for every  $\mathbf{b} \in \mathbb{R}^m$
  - · Every  $\mathbf{b} \in \mathbb{R}^m$  can be written as a linear combination of the columns of  $\mathbf{A}$
  - · The set of *m*-many hyper-planes simultaneously intersect for at least one point in  $\mathbf{R}^n$



- Underdetermined Systems : If m < n for  $\mathbf{A} \in \mathbb{R}^{m \times n}$  then we say the system is underdetermined.
  - · If an underdetermined system has a solution then this solution cannot be unique.
- Overdetermined Systems : If m > n for  $\mathbf{A} \in \mathbb{R}^{m \times n}$  then we say the system is overdetermined.
  - · An overdetermined system cannot have a solution for every  $\mathbf{b} \in \mathbb{R}^m$ .
- Square Systems : If m = n for  $\mathbf{A} \in \mathbb{R}^{m \times n}$  then we say the system is square.
  - · It is possible for a square system to have a unique solution for every  $\mathbf{b} \in \mathbb{R}^n$ .

Square systems are particularly interesting since there is the possibility of existence and uniqueness for every  $\mathbf{b} \in \mathbb{R}^n$ . The following statements are equivalent and, when taken together, is typically called the invertible matrix theorem (IMT):

- $\cdot$  A is an invertible matrix
- $\cdot$  **A**<sup>-1</sup> exists
- $\cdot$  A  $\sim$  I
- · A solution to Ax = b exists and this solution is  $x = A^{-1}b$
- $\cdot$  The hyper-planes intersect at only one point in  $\mathbb{R}^n$
- $\cdot$  Every  ${\bf b} \in \mathbb{R}^n$  can be written as a linear combination of the columns from  ${\bf A}$
- $\cdot \det(\mathbf{A}) \neq 0$

Since not every system will have square coefficient data a more general construct is required to discuss solubility of linear systems. This construct will use the following definitions:

Linear Combination : We say that b ∈ ℝ<sup>m</sup> is a linear combination of vectors from S = {a<sub>1</sub>, a<sub>2</sub>, a<sub>3</sub>, ..., a<sub>n</sub>} if there exist scalars x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>, ..., x<sub>n</sub> such that,

$$\mathbf{b} = \sum_{i=1}^{n} x_i \mathbf{a}_i = x_i \mathbf{a}_i = \mathbf{A}\mathbf{x}$$
(1)

· The set of all such **b** defines the spanning set of S

$$span(S) = \left\{ \mathbf{b} \in \mathbb{R}^m : \mathbf{b} = \sum_{i=1}^n x_i \mathbf{a}_i \right\}$$
(2)

Of the vectors in *S* it is possible that not all of them are needed to represent every element in span(S). In order to determine those vectors from *S* needed to represent every element in span(S) we define:

Linear Independence : We say that a set of vectors
S = {v<sub>1</sub>, v<sub>2</sub>, v<sub>3</sub>, ..., v<sub>k</sub>} forms a linearly independent set if and only if

$$\mathbf{0} = \sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{V}\mathbf{C},\tag{3}$$

has only the trivial solution  $\mathbf{c} = \mathbf{0}$ .

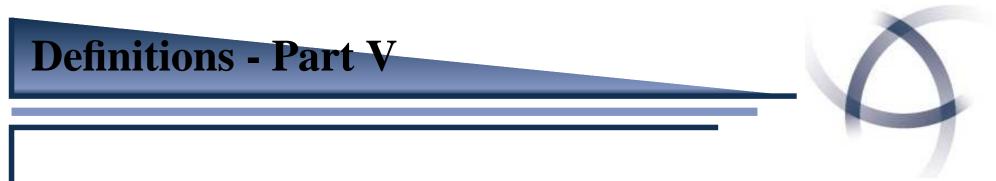
Spanning sets are an example of a so-called 'vector-space' and since every nontrivial spanning set contains an infinite amount amount of vectors we must define some way to characterize different spanning sets. To do this we define the following:

- <u>Basis</u> : Given a spanning space we define a basis for this space to be any collection of linearly independent vectors, from the space, that spans the space.
- <u>Dimension</u>: Given a spanning space we define the dimension of this space to be the number of vectors in any basis for this space.

Key Point: Any two vector spaces, which have the same dimension are structurally identical.

The previous terminology now allows us to define the following spaces, which are important to the study of Ax = b.

- Null-Space : Given  $\mathbf{A} \in \mathbb{R}^{m \times n}$  we define the null-space of  $\overline{\mathbf{A}}$ , Nul( $\mathbf{A}$ ), to be the set of all solutions to  $\mathbf{A}\mathbf{x} = \mathbf{0}$ .
  - The null-space is a vector space.
  - The null-space describes how the linear objects simultaneously intersect.
  - A basis for the null-space is found by explicitly solving the homogeneous equation.
  - The dimension of the null-space is the number of free-variables in the system.



- Column-Space : Given  $\mathbf{A} \in \mathbb{R}^{m \times n}$  we define the column-space of  $\mathbf{A}$ , Col( $\mathbf{A}$ ), to be the space of all linear combinations of the columns of  $\mathbf{A}$ .
  - · The column-space is a vector space.
  - · The column-space describes the set of all inhomogeneous vectors for which Ax = b is solvable.
  - A basis for the column-space is the set of linearly independent columns from **A**.
  - The dimension of the column-space is the number of linearly independent columns of **A**.
  - Since row-reduction does not change the solution to a linear system it does not change dependency relations. That is, if Ax = 0 has only the trivial solution and A ~ B then Bx = 0 has only the trivial solution.



- Row-Space : Given  $A \in \mathbb{R}^{m \times m}$  we define the row-space of  $\overline{A}$ , Row(A), to be the space of all linear combinations of the rows of A.
  - A basis for the row-space is the set of linearly independent rows from **A**.
  - The dimension of the row-space is the number of linearly independent rows of **A**.
  - · Row-reduction is reversible. That is, if  $\mathbf{A} \sim \mathbf{B}$  then the linearly independent rows of  $\mathbf{B}$  can be used as a basis for  $Row(\mathbf{A})$ .

Using the idea of vector spaces we now have the following statements, which completely characterizes the general linear problem  $\mathbf{A}_{m \times n} \mathbf{x}_{n \times 1} = \mathbf{b}_{m \times 1}$ :

- · Ax = b is solvable if and only if  $b \in Col(A)$
- · If Ax = b is solvable and the dimension of Nul(A) is zero then Ax = b is uniquely solvable.
- · Rank-Nullity Theorem: Rank  $\mathbf{A} + \dim(\operatorname{Nul}(\mathbf{A})) = n$  where Rank  $\mathbf{A}=\dim(\operatorname{Col}(\mathbf{A}))=\dim(\operatorname{Row}(\mathbf{A}))$ .

With this framework we can make the following additions to the IMT:

- $\cdot \mathbf{A}_{n \times n}$  is an invertible matrix
- $\cdot \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n\} = \mathbb{R}^n$
- · A basis for the column-space of **A** is a basis for  $\mathbb{R}^n$
- $\cdot \operatorname{Rank} \mathbf{A} = n$
- $\cdot$  The dimension of the null-space of  $\boldsymbol{A}$  is zero
- $\cdot$  **Ax** = **0** has only the trivial solution