MATH 332 - Linear Algebra July 28, 2009
Homework 8, Summer 2009 **Due**: August 3, 2009

Inner-Product - Norm - Orthogonality - Gram-Schmidt - QR Factorization

1. Let,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1}$$

Determine $\mathbf{U}$, associated with the similarity transformation $\sigma_z = \mathbf{U}\sigma_x\mathbf{U}^{\mathrm{T}}$ where $\mathbf{U}$ is an orthogonal matrix. [1]

2. Prove the following:

   (a) Let $\mathbf{u}$,$\mathbf{v} \in \mathbb{R}^n$. Prove that $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$.

   (b) Let W be a subspace of $\mathbb{R}^n$. Prove that $\mathrm{W}^{\perp}$ is a subspace of $\mathbb{R}^n$. [2]

   (c) Let $\mathbf{U}$ be an orthogonal matrix. Prove that $||\mathbf{U}\mathbf{x}|| = ||\mathbf{x}||$.

   (d) Let $\mathbf{U}_{n\times n}$ be an orthogonal matrix and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that $\mathbf{U}\mathbf{x}\cdot\mathbf{U}\mathbf{y} = \mathbf{x}\cdot\mathbf{y}$.

   (e) Let $\mathbf{U}_{n\times n}$ be an orthogonal matrix and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that $\mathbf{U}\mathbf{x}\cdot\mathbf{U}\mathbf{y} = 0$ if and only if $\mathbf{x}\cdot\mathbf{y} = 0$.

3. Given,

$$\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

   (a) Let $\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2]$. Compute $\mathbf{U}^{\mathrm{T}}\mathbf{U}$ and $\mathbf{U}\mathbf{U}^{\mathrm{T}}$.

   (b) Let $\mathrm{W} = \mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Compute $\mathrm{proj}_{\mathrm{W}}\,\mathbf{y}$ and $(\mathbf{U}\mathbf{U}^{\mathrm{T}})\mathbf{y}$.

   (c) Write $\mathbf{y}$ as the sum of a vector $\hat{\mathbf{y}}$ in W and a vector $\mathbf{z}$ in $\mathrm{W}^{\perp}$.

   (d) Describe the geometric relationship between the plane W in $\mathbb{R}^3$ and the vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ from part $\mathbf{c}$.

4. Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}.$$

Determine the $\mathbf{QR}$ factorization of $\mathbf{A}$.

5. In homework 6 we showed that the first four Hermite polynomials were linearly independent and thus a basis for $\mathbb{P}_3$.[3] While this makes good use of the material from 4.4 outside of the context of $\mathbb{R}^n$ it really misses the point.[4] The Hermite polynomials are orthogonal polynomials and constitute an orthonormal basis for vector space $L^2(-\infty,\infty)$.[5] To see why this is true we must define the inner-product to be,

$$f\cdot g = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}dx, \tag{2}$$

[1] Consider diagonalization of σ_x and notice that its eigenvectors are orthogonal. One could hope that these might provide the columns of the $\mathbf{U}$ matrix if they were unit-length.

[2] The procedure for this proof is outlined in Lay (problem 6.1.30 pg.383).

[3] We take without proof that the first $n+1$ Hermite polynomials are linearly independent and thus a basis for $\mathbb{P}_n$.

[4] The Hermite polynomials are prevalent in statistics, applied mathematics and physics but not in the context of polynomial spaces.

[5] The vector space $L^2(-\infty,\infty)$ is an infinite dimensional complete inner-product space or a Hilbert space, in honor of David Hilbert `http://en.wikipedia.org/wiki/David_Hilbert`. The space L^2, which is an abstraction of standard Euclidean space, is important because its elements must have finite length and any infinite-sequence of elements must converge to a point in L^2. The condition that 'vectors' must have finite length typically implies that they have finite energy, which is what one would hope. While, the convergence properties allows use to take limits without leaving the space.[6]

which is different than our standard definition in $\mathbb{R}^n$. [8] We take without proof that this definition satisfies the axioms of an inner-product. Recall the first few Hermite Polynomials, [9]

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = -2 + 4x^2, \quad H_3(x) = -12x + 8x^3, \quad x \in (-\infty, \infty),$$

satisfying the Rodrigues representation,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{3}$$

(a) Prove that $H_{2n}(x)$ is and even function and that $H_{2n+1}(x)$ is an odd function. [10]

(b) Prove that the even Hermite polynomials are orthogonal to the odd Hermite polynomials.

(c) Normalize H_0 and H_1.

(d) Using the normalized Hermite polynomials apply Gram-Schmidt and find $H_2(x)$. [11]

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[6] Indeed, things would be very bad if this were not the case. Consider the infinite sum, $\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$. The summands are all rational but this sum converges to π, which is irrational. That is, the rationals are not closed under limits of arbitrary linear combinations![7]

[7] Yeah, I footnoted a footnote. What of it?!

[8] If we used the standard inner-product and made the Hermite polynomials an orthonormal basis, via GramSchmidt, for $\mathbb{P}_n$ then we would have gotten to the standard polynomial basis, which is nothing new.

[9] For more we can look at `http://en.wikipedia.org/wiki/Hermite_polynomials`. There are, in general, infinitely-many of them arising as eigenfunctions of the differential operator $\frac{d^2}{dx^2} - x\frac{d}{dx}$.

[10] Recall that an even function has the property that $f(-x) = f(x)$ and an odd function has the property that $f(-x) = -f(x)$. To make this clear from Rodrigues representation you should show that the derivative of an even function is an odd function and that the derivative of an odd function is an even function.

[11] MIT's open courseware site has a nice discussion of GS applied to the Legendre polynomials. `web.mit.edu/18.06/www/Spring09/legendre.pdf` To do this first consider a general quadratic, $H_2(x) = ax^2 + bx + c$, and argue that $b = 0$. Next, we want to find a and c such that $H_2(x)$ is orthogonal to $H_1(x)$ and $H_0(x)$. Gram-Schmidt gives us a formula for this, page 404 of the text, only every inner-product must be thought of in the sense of (2). After this calculation you should have a relation between a and c. To find 'a' normalize $H_2(x)$ and compare your result to $H_2(x)$ as it is given. They should look the same up a multiplicative constant.