EM waves revisited

Review of EM wave equation and plane waves

Energy and intensity in EM waves

Maxwell's Equations to wave eqn

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla} \cdot \mathbf{E} = 0 \qquad \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0 \qquad \vec{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}$$

Take the curl:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = -\frac{\partial}{\partial t}\vec{\nabla} \times \mathbf{B} = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}\right)$$

Use the vector ID:

$$\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right) = \mathbf{B}\left(\mathbf{A} \cdot \mathbf{C}\right) - \mathbf{C}\left(\mathbf{A} \cdot \mathbf{B}\right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \mathbf{E}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\mathbf{E} = -\vec{\nabla}^2 \mathbf{E}$$

$$\boxed{\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}}$$

"Inhomogeneous Wave Equation"

Wave equation in a medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- The polarization (**P**) is sometimes a driving term for the waves (esp in nonlinear optics). In this case the polarization determines which frequencies will occur.
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}$$

- In nonlinear optics, the induced polarization is more complicated:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \left(\chi^{(1)} \mathbf{E} + \chi^{(2)} \mathbf{E}^2 + \chi^{(3)} \mathbf{E}^3 + \ldots \right)$$

- The extra nonlinear terms can lead to new frequencies.

Solving the wave equation: linear induced polarization

For low irradiances, the polarization is proportional to the incident field:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}$$

The displacement vector combines the effect of E and P

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1+\chi)\mathbf{E} = \varepsilon \mathbf{E} = n^2 \mathbf{E}$$

In this simple (and most common) case, the wave equation becomes:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2}\chi \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \rightarrow \vec{\nabla}^2 \mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Using: $\varepsilon_0 \mu_0 = 1/c^2$ $\qquad$ $\varepsilon_0 (1+\chi) = \varepsilon = n^2$

D does not always point in the same direction as **E**:

- Birefringence: n is a function of linear polarization direction
- Optically activity: n is a function of circular polarization state
- In this case, χ and ε are tensors operating on **E** to change its direction

Vector wave equation

- The EM wave equation in vector form

$$\vec{\nabla}^2\mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

- **E** is a vector that depends on **r** and t
- Wave equation is actually a set of 3 equations:

$$\vec{\nabla}^2 E_x(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}E_x(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_y(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}E_y(\mathbf{r},t) = 0$$

$$\vec{\nabla}^2 E_z(\mathbf{r},t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}E_z(\mathbf{r},t) = 0$$

Plane wave solutions for the wave equation

If we assume the solution has no dependence on x or y:

$$\vec{\nabla}^2\mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t)$$

$$\rightarrow \frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

The solutions are oscillating functions, for example

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(k_z z - \omega t)$$

Where $\omega = k\,c, \quad k = 2\pi n / \lambda, \quad v_{ph} = c / n$

This is a linearly polarized wave.

Complex notation for waves

- Write cosine in terms of exponential

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t + \phi) = \hat{\mathbf{x}} E_x \frac{1}{2}\left(e^{i(kz-\omega t+\phi)} + e^{-i(kz-\omega t+\phi)}\right)$$

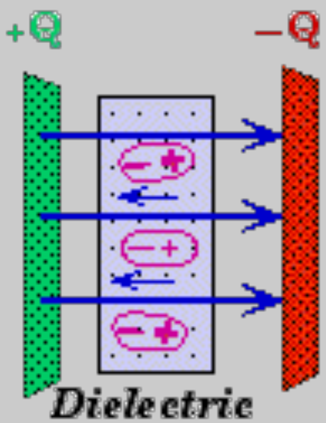
  - Note E-field is a *real* quantity.
  - It is convenient to work with just one part
    - We will use $E_0 e^{+i(kz-\omega t)}$
    - Svelto: $E_0 e^{-i(kz-\omega t)}$
  - Then take the real part.
    - No factor of 2
    - In *nonlinear* optics, we have to explicitly include conjugate term

Wave energy and intensity

- Both E and H fields have a corresponding energy density (J/m^3)
  - For static fields (e.g. in capacitors) the energy density can be calculated through the work done to set up the field

    $$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2$$

  - Some work is required to polarize the medium
  - Energy is contained in both fields, but H field can be calculated from E field



Calculating H from E in a plane wave

- Assume a non-magnetic medium

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t)$$

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

  – Can see **H** is perpendicular to **E**

$$-\mu_0 \frac{\partial \mathbf{H}}{\partial t} = \vec{\nabla} \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \hat{\mathbf{y}} \partial_z E_x = -\hat{\mathbf{y}} k E_x \sin(kz - \omega t)$$

  – Integrate on time to get **H**-field:

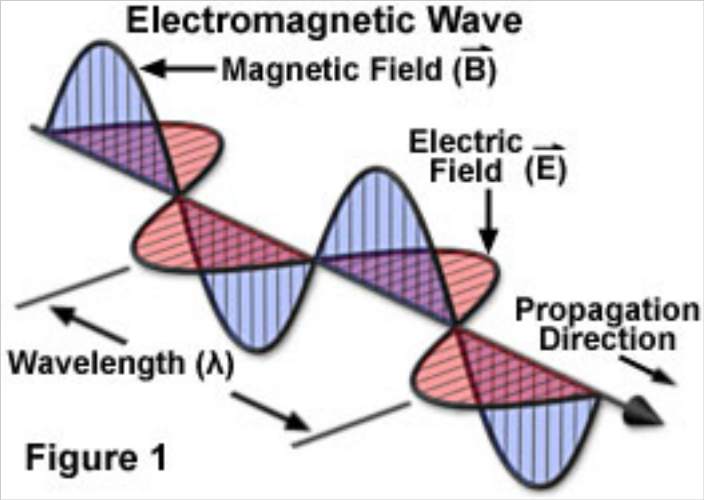
$$\mathbf{H} = \hat{\mathbf{y}} \int \frac{k E_x}{\mu_0} \sin(kz - \omega t) dt = \hat{\mathbf{y}} \frac{k E_x}{\mu_0} \left(\frac{-\cos(kz - \omega t)}{-\omega} \right) = \hat{\mathbf{y}} \frac{k E_x}{\omega \mu_0} \cos(kz - \omega t)$$

H field from E field

- H field for a propagating wave is *in phase* with E-field

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t)$$

$$\mathbf{H}(z,t) = \hat{\mathbf{y}} \frac{k E_x}{\omega \mu_0} \cos(kz - \omega t)$$



- Amplitudes are not independent

$$H_y = \frac{k}{\omega \mu_0} E_x \qquad k = n\frac{\omega}{c} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0} \rightarrow \frac{1}{\mu_0 c} = \varepsilon_0 c$$

$$H_y = \frac{n}{c\mu_0} E_x = n\varepsilon_0 c E_x$$

Energy density in a traveling EM wave

- Back to energy density, non-magnetic

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu_0 H^2$$

$$H = n\varepsilon_0 c E$$

$$\varepsilon = \varepsilon_0 n^2$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\mu_0 n^2 \varepsilon_0{}^2 c^2 E^2$$

$$\mu_0 \varepsilon_0 c^2 = 1$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\varepsilon_0 n^2 E^2 = \varepsilon_0 n^2 E_0{}^2 \cos^2\left(kz - \omega t\right)$$

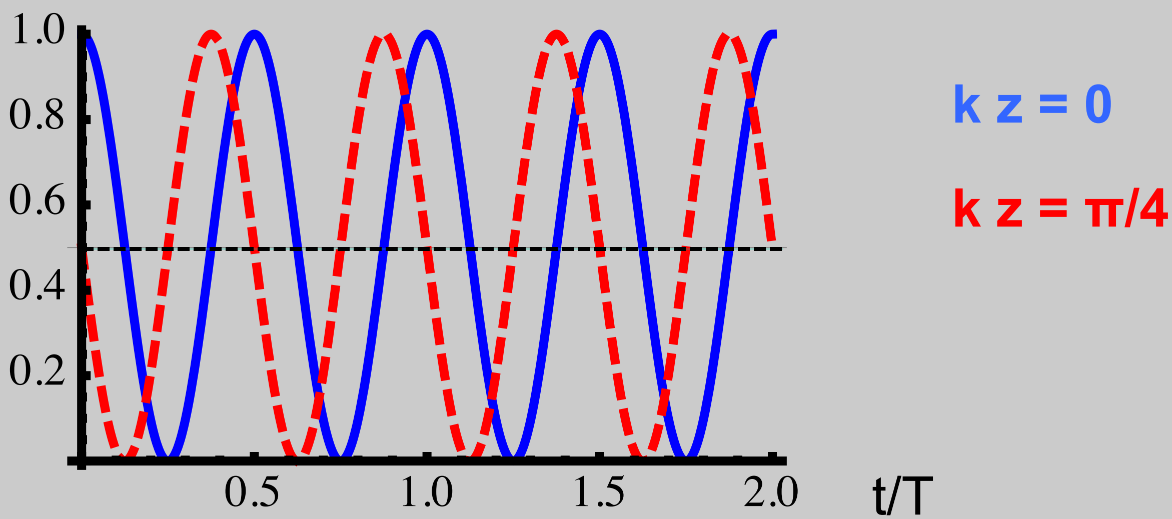
Equal energy in both components of wave

Cycle-averaged energy density

- Optical oscillations are faster than detectors
- Average over one cycle:

$$\langle \rho \rangle = \varepsilon_0 n^2 E_0{}^2 \frac{1}{T}\int_0^T \cos^2\left(k_z z - \omega t\right) dt$$

  - Graphically, we can see this should = ½



  - Regardless of position z

$$\langle \rho \rangle = \frac{1}{2}\varepsilon_0 n^2 E_0{}^2$$

Intensity and the Poynting vector

- Intensity is an energy flux (J/s/cm^2)
- In EM the Poynting vector give energy flux

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

  - For our plane wave,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = E_0 \cos\left(k_z z - \omega t\right) n\varepsilon_0 c E_0 \cos\left(k_z z - \omega t\right) \hat{\mathbf{x}} \times \hat{\mathbf{y}}$$

$$\mathbf{S} = n\varepsilon_0 c E_0^2 \cos^2\left(k_z z - \omega t\right) \hat{\mathbf{z}}$$

  - **S** is along **k**

- Time average: $\mathbf{S} = \tfrac{1}{2} n\,\varepsilon_0 c E_0^2 \hat{\mathbf{z}}$
- *Intensity* is the magnitude of **S**

$$I = \frac{1}{2} n\varepsilon_0 c E_0^2 = \frac{c}{n}\rho = V_{phase} \cdot \rho$$

Photon flux: $F = \dfrac{I}{h\nu}$

General 3D plane wave solution

- Assume separable function

$$\mathbf{E}(x,y,z,t) \sim f_1(x) f_2(y) f_3(z) g(t)$$

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(z,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} = \mathbf{E_0} e^{i\left(k_x x + k_y y + k_z z\right)} e^{-i\omega t}$$

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

  – Now k-vector can point in arbitrary direction

- With this solution in W.E.:

$$n^2 \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}$$

Valid even in waveguides and resonators

Grad and curl of 3D plane waves

- Simple trick:

$$\nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z$$

  – For a plane wave,

$$\nabla \cdot \mathbf{E} = i\left(k_x E_x + k_y E_y + k_z E_z\right) = i\left(\mathbf{k} \cdot \mathbf{E}\right)$$

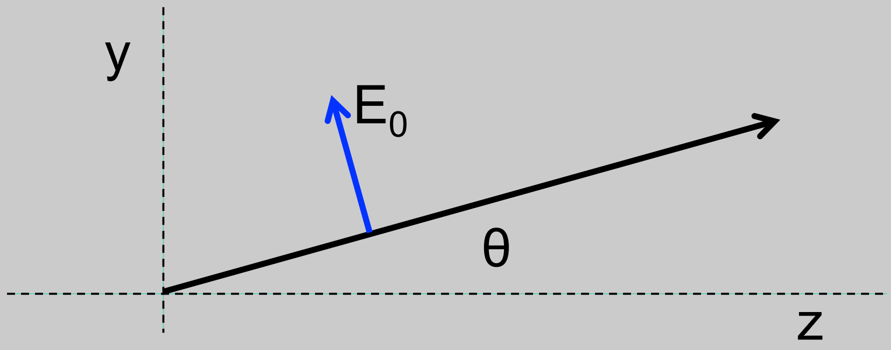
  – Similarly,

$$\nabla \times \mathbf{E} = i\left(\mathbf{k} \times \mathbf{E}\right)$$

- Consequence: since $\nabla \cdot \mathbf{E} = 0, \quad \mathbf{k} \perp \mathbf{E}$
  – For a given **k** direction, **E** lies in a plane
  – E.g. **x** and **y** linear polarization for a wave propagating in **z** direction

Writing electric field expressions

- Write an expression for a complex E-field as shown:



$$\mathbf{E}(y,z,t) = E_0\left[\hat{\mathbf{y}}\cos\theta - \hat{\mathbf{z}}\sin\theta\right]e^{i(ky\sin\theta + kz\cos\theta - \omega t)}$$