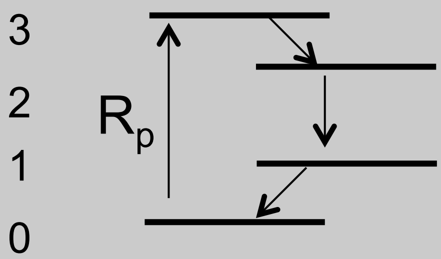
Modeling laser dynamics

- Energy input and output:
  - Input pump energy
  - Output power
  - Waste heat
- Internal *energy* balance:
  - Stored energy in inversion density
  - Circulating beam power
- Internal *number* balance:
  - Number of atoms inverted
  - Number of photons in cavity

Equations for lasing dynamics

- Consider a 4-level system:



Assume: τ_{32} and $\tau_{10} << \tau_{21}$ and $W_{21}N_2$

- First look at population dynamics for level 2

$$\frac{dN_2}{dt} = R_P - W\,N_2 - N_2/\tau_{21}$$

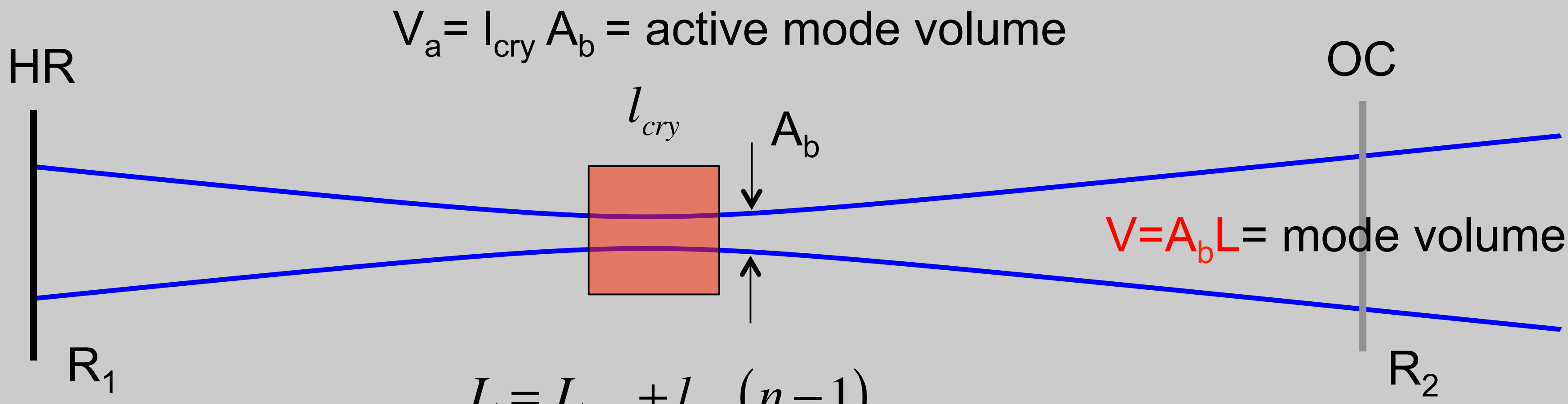
$R_p =$ Pump rate/volume

  – W is the stimulated emission rate

$$W = \rho B_{21} = \frac{2I}{c}\frac{c\sigma_{21}}{h\nu_{21}} = \frac{2\sigma_{21}}{h\nu_{21}} I$$

- We will want to keep track of two variables:
  - N_2: population inversion density (#/vol)
  - ϕ: total number of photons in cavity mode

Model laser cavity



$$L = L_{cav} + l_{cry}(n-1)$$

$\mathcal{L}_i$ Internal passive losses

$$T_{RT} = \frac{2L}{c}$$ Round-trip time

$$\phi = \frac{\rho V}{h\nu_{21}}$$ Total number of photons in cavity

$$\rho = \frac{2I}{c}$$ Energy density. 2 I = average intensity (both directions), neglecting standing wave interference

$$\phi = \frac{2I}{c}\frac{L A_b}{h\nu_{21}} = I\frac{A_b T_{RT}}{h\nu_{21}}$$

Rate equation for population

- Represent stimulated emission in terms of ϕ
  - Let $W = B\phi$ where *B* is the stimulated emission rate per photon

Energy density

Einstein B

$$W = \rho B_{21} \quad \text{with} \quad \rho = \frac{\phi h\nu_{21}}{V} \quad \text{and} \quad B_{21} = \frac{c\sigma_{21}}{h\nu_{21}}$$

$$W = \frac{\phi h\nu_{21}}{V}\frac{c\sigma_{21}}{h\nu_{21}} = B\phi \rightarrow B = \frac{c\sigma_{21}}{V}$$

Cavity mode volume

$$V = A_b L$$

$$B = \frac{c}{A_b L}\sigma_{21} \rightarrow \frac{2\sigma_{21}}{A_b T_{RT}}$$

Round trip time

$$T_{RT} = \frac{2L}{c}$$

$$\frac{dN_2}{dt} = R_P - B\phi N_2 - N_2 / \tau_{21}$$

Photon gain rate

- Since *B* is the stimulated emission rate per photon

$$\frac{d\phi}{dt} = BV_a N_2 \phi$$

$$V_a N_2 = \text{number of atoms capable of contributing to beam}$$

- Alternative: calculate power emitted per unit volume of crystal

$$\frac{dP}{dV_a} = B_{21}\rho N_2 h\nu_{21} \qquad \frac{dP}{dV_a} = \frac{V}{V_a}\frac{d\rho}{dt} = \frac{h\nu_{21}}{V_a}\frac{d\phi}{dt} \qquad \text{with} \qquad \rho = \frac{\phi}{V}h\nu_{21}$$

$$\frac{d\phi}{dt} = B_{21}\frac{\phi}{V}h\nu_{12}V_a N_2 \qquad \text{with} \qquad B_{21} = \frac{c\sigma_{21}}{h\nu_{21}}$$

$$\frac{d\phi}{dt} = \frac{c\sigma_{21}}{h\nu_{21}}\frac{\phi}{V}h\nu_{12}V_a N_2 = \frac{\sigma_{21}c}{V}V_a N_2 \phi \qquad \text{with} \qquad B = \frac{\sigma_{21}c}{V}$$

$$\rightarrow \frac{d\phi}{dt} = BV_a N_2 \phi$$

Passive cavity loss rate

- Light losses in cavity from
  - mirrors (output coupling and leakage)
  - internal losses (reflections, absorption, scatter, misalignment)
- Set up differential equation
  - Assume losses are small, can be approximated as evenly distributed
  - After *m* passes through gain medium: $t_m = mT_{RT}/2$

$$\phi(t_m) = \left[R_1 R_2 (1-\mathcal{L}_i)^2 \right]^{m/2} \phi_0$$

$$\rightarrow \phi(t_m) = e^{-m\gamma}\phi_0$$

Define γ so that

$$e^{-\gamma} \equiv \sqrt{R_1 R_2}(1-\mathcal{L}_i)$$

$$\gamma = -\ln\left[\sqrt{R_1 R_2}(1-\mathcal{L}_i)\right]$$

$$\frac{\phi(t_m)-\phi(t_{m-1})}{T_{RT}/2} = \frac{e^{-\gamma}\phi_m - \phi_{m-1}}{T_{RT}/2} \rightarrow \frac{d\phi}{dt} = -\left(\frac{1-e^{-\gamma}}{T_{RT}/2}\right)\phi \approx -\frac{2\gamma}{T_{RT}}\phi$$

$$\frac{d\phi}{dt} = -\frac{\phi}{\tau_c}$$

$$\tau_c = T_{RT}/2\gamma = L/\gamma c$$

Photon cavity lifetime

Laser dynamics equations

- Combine gain and loss terms for photon number

$$\frac{d\phi}{dt} = V_a B N_2 \phi - \frac{\phi}{\tau_c}$$

$$\frac{dN_2}{dt} = R_P - B\phi N_2 - N_2 / \tau_{21}$$

- Output power: from mirror M_2

$$\text{OC transmission} = 1 - R_2$$

$$\text{output power} = \left(1 - R_2\right) \cdot \text{intracavity power}$$

$$P_{out} = \gamma_2 \frac{h\nu_{21}}{T_{RT}} \phi = \gamma_2 \frac{c}{2L} h\nu_{21} \phi$$

Can add a photon for vacuum contribution

$$\frac{d\phi}{dt} = V_a B N_2 \left(\phi + 1\right) - \frac{\phi}{\tau_c}$$

this allows build-up to get started

Separate loss terms:

$$\gamma = \gamma_i + \tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2$$

Lasing threshold

- At threshold, gain balances loss

$$\frac{d\phi}{dt} = \left(V_a B N_2 - \frac{1}{\tau_c} \right) \phi \qquad V_a B N_2 \geq \frac{1}{\tau_c} \quad \text{for net gain}$$

  - Critical inversion density N_c

$$\tau_c = L / \gamma c$$

$$N_c = \frac{1}{V_a B \tau_c} = \frac{1}{V_a} \frac{V}{\sigma_{21} c} \frac{\gamma c}{L} = \frac{L}{l_{cry}} \frac{1}{\sigma_{21}} \frac{\gamma}{L} \qquad B = \frac{\sigma_{21} c}{V}$$

$$\boxed{N_c = \frac{\gamma}{\sigma_{21} l_{cry}}}$$

  - At threshold, $\begin{matrix} \phi \approx 0 \\ N_2 = N_c \end{matrix}$ $\quad \dfrac{dN_2}{dt} = 0 = R_{cp} - N_c / \tau_{21}$

  - Critical pumping rate: $\boxed{R_{cp} = N_c / \tau_{21} = \gamma / \sigma_{21} l_{cry} \tau_{21}}$

Lasing above threshold

- pumping rate exceeds the critical value
- Steady state: time derivatives = 0
- Find steady-state values: N_0 and ϕ_0

$$\frac{d\phi}{dt} = 0 = V_a B N_0 \phi_0 - \frac{\phi_0}{\tau_c} \rightarrow \left(V_a B N_0 - \frac{1}{\tau_c} \right) \phi_0 = 0$$

$$\therefore N_0 = \frac{1}{V_a B \tau_c} = N_{th}$$

$$\frac{dN_2}{dt} = 0 = R_P - B\phi_0 N_0 - N_0 / \tau_{21} \rightarrow \phi_0 = \frac{R_P - N_0 / \tau_{21}}{B N_0}$$

$$\phi_0 = \tau_c \left(V_a R_P - \frac{V_a N_0}{\tau_{21}} \right) = \text{cavity storage time} \left(\text{atom pump rate - fluor loss rate} \right)$$

Intracavity photon number vs pumping

- ϕ_0 is the steady-state circulating photon number in the cavity
- Rearrange to relate R_p to critical pump rate R_{cp}

$$\phi_0 = V_a \tau_c \left(R_P - N_0 / \tau_{21} \right) \qquad R_{cp} = N_c / \tau_{21} \qquad N_0 = N_c$$

$$\phi_0 = V_a \tau_c R_{cp} \left(\frac{R_P}{R_{cp}} - 1 \right) \qquad R_{cp} = \gamma / \sigma_{21} l_{cry} \tau_{21}$$

$$\frac{R_P}{R_{cp}} = \frac{P_p}{P_{th}}$$

$$\phi_0 = \frac{V_a \tau_c \gamma}{\sigma_{21} l_{cry} \tau_{21}} \left(\frac{P_p}{P_{th}} - 1 \right) = \frac{A_b}{\sigma_{21}} \frac{\tau_c}{\tau_{21}} \gamma \left(\frac{P_p}{P_{th}} - 1 \right)$$

Output power and slope efficiency

- Mirror M_2 lets photons out:

$$P_{out} = \gamma_2 \frac{h\nu_{21}}{T_{RT}} \phi_0$$

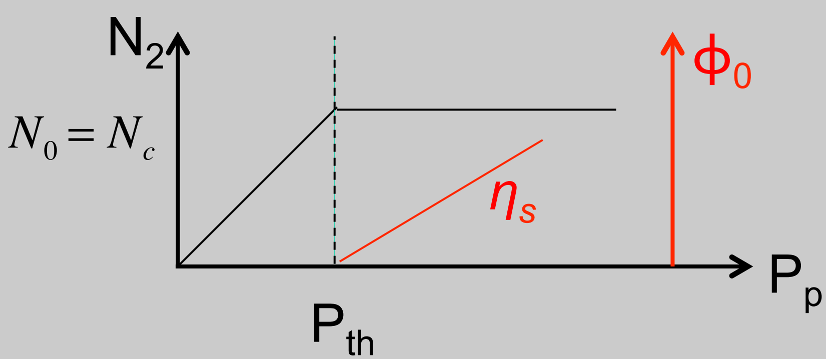
$$\phi_0 = \frac{A_b}{\sigma_{21}} \frac{\tau_c}{\tau_{21}} \gamma \left(\frac{P_p}{P_{th}} - 1 \right)$$

$$P_{out} = \gamma_2 \frac{h\nu_{21}}{T_{RT}} \frac{A_b}{\sigma_{21}} \frac{\tau_c}{\tau_{21}} \gamma \left(\frac{P_p}{P_{th}} - 1 \right)$$

$$I_s = \frac{h\nu_{21}}{\sigma_{21}\tau_{21}}$$

$$P_{out} = I_s A_b \frac{\gamma_2}{2} \left(\frac{P_p}{P_{th}} - 1 \right)$$

$$\tau_c = T_{RT} / 2\gamma$$



Slope efficiency: $\eta_s = \frac{I_s A_b}{P_{th}} \frac{\gamma_2}{2}$

At first pump power goes into reaching threshold,
then extra pump goes into output

Pump threshold power

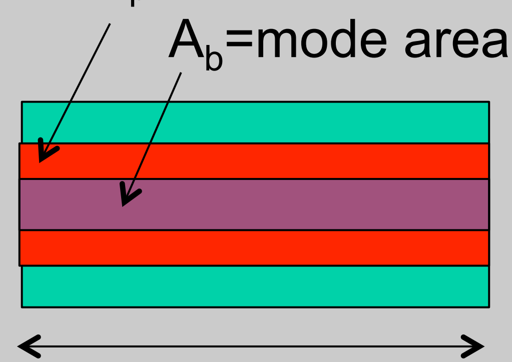
- Critical pumping rate $$R_{cp} = \frac{\gamma}{\sigma_{21} l_{cry}} \frac{1}{\tau_{21}}$$

- Connect pump rate to pump power

$$R_p = \eta_p \frac{P}{h\nu_p} \frac{1}{A_p l_{cry}}$$

← Volume of pump absorption

↖ Photon arrival rate



$$R_{cp} = \frac{\gamma}{\sigma_{21} l_{cry}} \frac{1}{\tau_{21}} = \eta_p \frac{P_{th}}{h\nu_p} \frac{1}{A_p l_{cry}}$$

$$\rightarrow P_{th} = \frac{\gamma}{\eta_p \sigma_{21}} \frac{h\nu_p}{\tau_{21}} A_p = I_s A_p \frac{\gamma}{\eta_p} \frac{h\nu_p}{h\nu_{21}}$$

Note ratio of photon energies of pump and lasing photons

Quantifying laser performance

- Measure output power vs input power
  - Get threshold power and slope efficiency

$$P_{th} = I_s A_p \frac{\gamma}{\eta_p} \frac{h\nu_p}{h\nu_{21}}$$

$$\eta_s = \frac{I_s A_b}{P_{th}} \frac{\gamma_2}{2} = \frac{I_s A_b}{I_s A_p \frac{\gamma}{\eta_p} \frac{h\nu_p}{h\nu_{21}}} \frac{\gamma_2}{2} = \frac{A_b}{A_p} \frac{h\nu_{21}}{h\nu_p} \eta_p \frac{\gamma_2}{2\gamma}$$

  - Principal unknown is the cavity loss, γ

Extensions of model

- Assumptions so far:
  - spatially uniform pumping, uniform beam profile
  - Fast depletion from level 1
  - Perfect, and fast transfer from pump band to level 2
- Other complications
  - 3-level and quasi- 3-level systems
  - Inhomogeneous broadening
  - Transients…