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Tensor character of  $\chi^{(3)}$   
Jones matrix method for polarization  
nonlinear ellipse rotation

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# Centro-symmetric media

- For second-order response, the potential must have asymmetry.
- When the binding potential for the electrons is centrally symmetric, the response can still be nonlinear, but the order must be odd (3<sup>rd</sup>, 5<sup>th</sup>, etc).
- Consider a central restoring force:

$$\mathbf{F}(\mathbf{r}) = -m\omega_0^2 \mathbf{r} + mb(\mathbf{r} \cdot \mathbf{r})\mathbf{r}$$

$$F_i(\mathbf{r}) = -m\omega_0^2 r_i + m b r_j r_j r_i$$

- force is always directed along  $\hat{\mathbf{r}}$  direction
- At large  $r$ , force is less binding.
- As with the non-centrosymmetric potential, perform perturbation expansion.
  - $x^{(2)}$  does not contribute, so  $\chi^{(2)}=0$

# Solution of 3<sup>rd</sup> order

- Each term for 1<sup>st</sup> order solution can be a different frequency

$$\ddot{x}^{(3)} + 2\gamma \dot{x}^{(3)} + \omega_0^2 x^{(3)} = b \left( x^{(1)} \right)^3$$

$$\left( \ddot{x}^{(3)}(\omega_q) + 2\gamma \dot{x}^{(3)}(\omega_q) + \omega_0^2 x^{(3)}(\omega_q) \right) e^{-i\omega_q t} = b \sum_{mnp} x^{(1)}(\omega_m) x^{(1)}(\omega_n) x^{(1)}(\omega_p) e^{-i(\omega_m + \omega_n + \omega_p)t}$$

- Note the m, n, p can all be + or - : for example,
- Enforce energy conservation, so  $\omega_{-2} = -\omega_2$  in summation

- Solution is

$$\omega_q = \omega_m + \omega_n + \omega_p \rightarrow (mnp)$$

$$\mathbf{r}^{(3)}(\omega_q) = - \sum_{(mnp)} \frac{be^3}{m^3} \frac{(\mathbf{E}(\omega_m) \cdot \mathbf{E}(\omega_n)) \mathbf{E}(\omega_p)}{D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}$$

$$\mathbf{P}^{(3)}(\omega_q) = -N e \mathbf{r}^{(3)}(\omega_q) = + \sum_{(mnp)} N \frac{be^4}{m^3} \frac{(\mathbf{E}(\omega_m) \cdot \mathbf{E}(\omega_n)) \mathbf{E}(\omega_p)}{D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}$$

# Calculation of $\chi^{(3)}$

- 3<sup>rd</sup> order NL polarization is

$$\mathbf{P}^{(3)}(\omega_q) = \sum_{(mnp)} N \frac{be^4}{m^3} \frac{(\mathbf{E}(\omega_m) \cdot \mathbf{E}(\omega_n)) \mathbf{E}(\omega_p)}{D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}$$

- Defined in terms of the susceptibility

$$P_i^{(3)}(\omega_q) \equiv \epsilon_0 \sum_{jkl} \sum_{(mnp)} \chi_{ijkl}^{(3)}(\omega_q, \omega_m, \omega_n, \omega_p) E_j(\omega_m) E_k(\omega_n) E_l(\omega_p)$$

- $\chi^{(3)}$  is a *tensor*:

- $i j k l$  are *coordinate indices* (1, 2, 3 or  $x, y, z$ ) that correspond to the directions of the *field* polarizations:  $i$  is output,  $j, k, l$  are distinct inputs
- $q, m, n, p$  are frequency indices of the distinct fields
- All indices can potentially be the same
- $(mnp)$  in summation means  $\omega_q = \omega_m + \omega_n + \omega_p$

# $\chi^{(3)}$ tensor

- Convert vector P to summation: e.g.

$$\mathbf{E}(\omega_m) \cdot \mathbf{E}(\omega_n) = \sum_j E_j(\omega_m) E_j(\omega_n) = \sum_{jk} E_j(\omega_m) E_k(\omega_n) \delta_{jk}$$

- 3<sup>rd</sup> order NL susceptibility is

$$P_i^{(3)}(\omega_q) = \sum_{jkl} \sum_{(mnp)} N \frac{be^4}{m^3} \frac{E_j(\omega_m) E_k(\omega_n) E_l(\omega_p) \delta_{jk} \delta_{il}}{D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}$$

$$P_i^{(3)}(\omega_q) \equiv \varepsilon_0 \sum_{jkl} \sum_{(mnp)} \chi_{ijkl}^{(3)}(\omega_q, \omega_m, \omega_n, \omega_p) E_j(\omega_m) E_k(\omega_n) E_l(\omega_p)$$

$$\chi_{ijkl}^{(3)}(\omega_q, \omega_m, \omega_n, \omega_p) = \frac{Nbe^4}{\varepsilon_0 m^3} \frac{\delta_{jk} \delta_{il}}{D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}$$

- Account for “intrinsic permutation symmetry”

- Fields  $E_j(\omega_m) E_k(\omega_n) E_l(\omega_p)$  can be in any order

$$\chi_{ijkl}^{(3)}(\omega_q, \omega_m, \omega_n, \omega_p) = \frac{Nbe^4}{3\varepsilon_0 m^3} \frac{\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}}{D(\omega_q) D(\omega_m) D(\omega_n) D(\omega_p)}$$

- 3 terms aren't there b/c of dot product of fields

# $\chi^{(3)}$ tensor: isotropic medium

- How can we simplify?

$$\begin{aligned} \mathbf{P}^{(3)}(\omega_q) &= -N \mathbf{e}^{(3)}(\omega_q) \quad \text{4th-rank tensor : 81 elements} \\ \Rightarrow \mathbf{P}_i^{(3)}(\omega_q) &= \sum_{jkl} \sum_{(mnp)} \underline{\chi_{ijkl}^{(3)}(\omega_q, \omega_m, \omega_n, \omega_p)} E_j(\omega_m) E_k(\omega_n) E_l(\omega_p) \\ &= D \sum_{jkl} \chi_{ijkl}^{(3)}(\omega_q, \omega_m, \omega_n, \omega_p) E_j(\omega_m) E_k(\omega_n) E_l(\omega_p) \end{aligned}$$

Element with even number of index

where, D : degeneracy factor

(The number of distinct permutations of the frequencies  $w_m, w_n, w_p$ )

Let's consider the 3<sup>rd</sup> order susceptibility for the case of an **isotropic material**.

21 nonzero elements :

$$\chi_{1111} = \chi_{2222} = \chi_{3333}$$

Rotation by 90° with all polarizations parallel

$$\chi_{1122} = \chi_{1133} = \chi_{2211} = \chi_{2233} = \chi_{3311} = \chi_{3322}$$

Rotation by 90° with pairs of polarization perpendicular

$$\chi_{1212} = \chi_{1313} = \chi_{2323} = \chi_{2121} = \chi_{3131} = \chi_{3232}$$

$$\chi_{1221} = \chi_{1331} = \chi_{2112} = \chi_{2332} = \chi_{3113} = \chi_{3223}$$

and,  $\chi_{1111} = \chi_{1122} + \chi_{1212} + \chi_{1221}$

# $\chi^{(3)}$ tensor depends on process

Express the nonlinear susceptibility in the compact form :

$$\underline{\underline{\chi_{ijkl} = \chi_{1122} \delta_{ij} \delta_{kl} + \chi_{1212} \delta_{ik} \delta_{jl} + \chi_{1221} \delta_{il} \delta_{jk}}}$$

Example) Third-harmonic generation :  $\chi_{ijkl}(3\omega=\omega+\omega+\omega)$

$$\begin{aligned} & \chi_{1122} = \chi_{1212} = \chi_{1221} \\ \Rightarrow & \chi_{ijkl}(3\omega=\omega+\omega+\omega) = \chi_{1122}(3\omega=\omega+\omega+\omega) \times (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \end{aligned}$$

Example) Intensity-dependent refractive index :  $\chi_{ijkl}(\omega=\omega+\omega-\omega)$

$$\begin{aligned} & \chi_{1122} = \chi_{1212} \neq \chi_{1221} \quad \textcolor{blue}{\omega_1 = \omega_2 \text{ but } \omega_3 = -\omega_1} \\ \Rightarrow & \chi_{ijkl}(\omega = \omega + \omega - \omega) = \chi_{1122}(\omega = \omega + \omega - \omega) \times (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl}) + \chi_{1221}(\omega = \omega + \omega - \omega) \delta_{il} \delta_{jk} \\ & \mathbf{P}_i(\omega) = 3 \sum_{jkl} \chi_{ijkl}(\omega = \omega + \omega - \omega) E_j(\omega) E_k(\omega) E_l(\omega) \\ \Rightarrow & \mathbf{P}_i(\omega) = 6 \chi_{1122} E_i(\mathbf{E} \cdot \mathbf{E}^*) + 3 \chi_{1221} E_i^*(\mathbf{E} \cdot \mathbf{E}) \quad \Rightarrow \quad \mathbf{P} = 6 \chi_{1122} (\mathbf{E} \cdot \mathbf{E}^*) \mathbf{E} + 3 \chi_{1221} (\mathbf{E} \cdot \mathbf{E}) \mathbf{E}^* \quad \text{in vector form} \end{aligned}$$

# Effective NL susceptibility

Defining the coefficients, A and B as

$$A=6\chi_{1122}, \quad B=6\chi_{1221} \quad (\text{Maker and Terhune's notation})$$

$$\boxed{\mathbf{P} = A(\mathbf{E} \cdot \mathbf{E}^*)\mathbf{E} + \frac{1}{2}B(\mathbf{E} \cdot \mathbf{E})\mathbf{E}^*}$$

$$P_i = \sum_j \chi_{ij} E_j$$

$$\Rightarrow \chi_{ij} = A'(\mathbf{E} \cdot \mathbf{E}^*)\delta_{ij} + \frac{1}{2}B'(E_i E_j^* + E_i^* E_j)$$

$$\text{where, } A' = A - \frac{1}{2}B = 6\chi_{1122} - 3\chi_{1221}$$

$$B' = B = 6\chi_{1221}$$

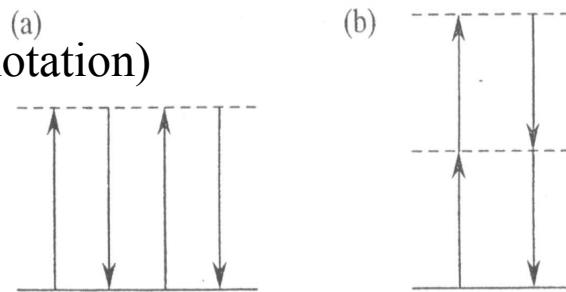


FIGURE 4.2.1 Diagrams (a) and (b) represent the resonant contributions to the non-linear coefficients A and B, respectively.

## Physical mechanisms

$B/A=6, \quad B'/A'=-3$  : molecular orientation

$B/A=1, \quad B'/A'=2$  : nonresonant electronic response

$B/A=0, \quad B'/A'=0$  : electrostriction

## 4.3 Nonresonant Electronic Nonlinearities

# The most fast response  $\tau = 2\pi a_0/v \approx 10^{-16} \text{ s}$  [ $a_0$ (Bohr radius)~0.5x10<sup>-8</sup>cm,  $v$ (electron velocity)~c/13]

Classical, Anharmonic Oscillator Model of Electronic Nonlinearities

$$\begin{aligned} \text{Approximated Potential } U(\mathbf{r}) &= \frac{1}{2} m \omega_0^2 |\mathbf{r}|^2 - \frac{1}{4} m b |\mathbf{r}|^4 \\ \chi_{ijkl}^{(3)}(\omega_q, \omega_m, \omega_n, \omega_p) &= \frac{Nbe^4 [\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]}{3m^3 D(\omega_q)D(\omega_m)D(\omega_n)D(\omega_p)} \quad (1.4.52) \end{aligned}$$

$$\Rightarrow \chi_{ijkl}^{(3)}(\omega = \omega + \omega - \omega) = \frac{Nbe^4 [\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]}{3m^3 D^3(\omega)D(-\omega)}$$

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$$\text{where, } D(\omega) = \omega_0^2 - \omega^2 - 2i\omega\gamma$$

According to the notation of Maker and Terhune,

$$A = B = \frac{2Nbe^4}{m^3 D^3(\omega)D(-\omega)}$$

Far off-resonant case  $\omega \ll \omega_0 \Rightarrow D(\omega) \approx \omega_0^2$ ,  $b \approx \omega_0^2/d^2$

$$\underline{\chi^{(3)} = \frac{Ne^4}{m^3 \omega_0^6 d^2}}$$

Typical value of  $\chi^{(3)}$

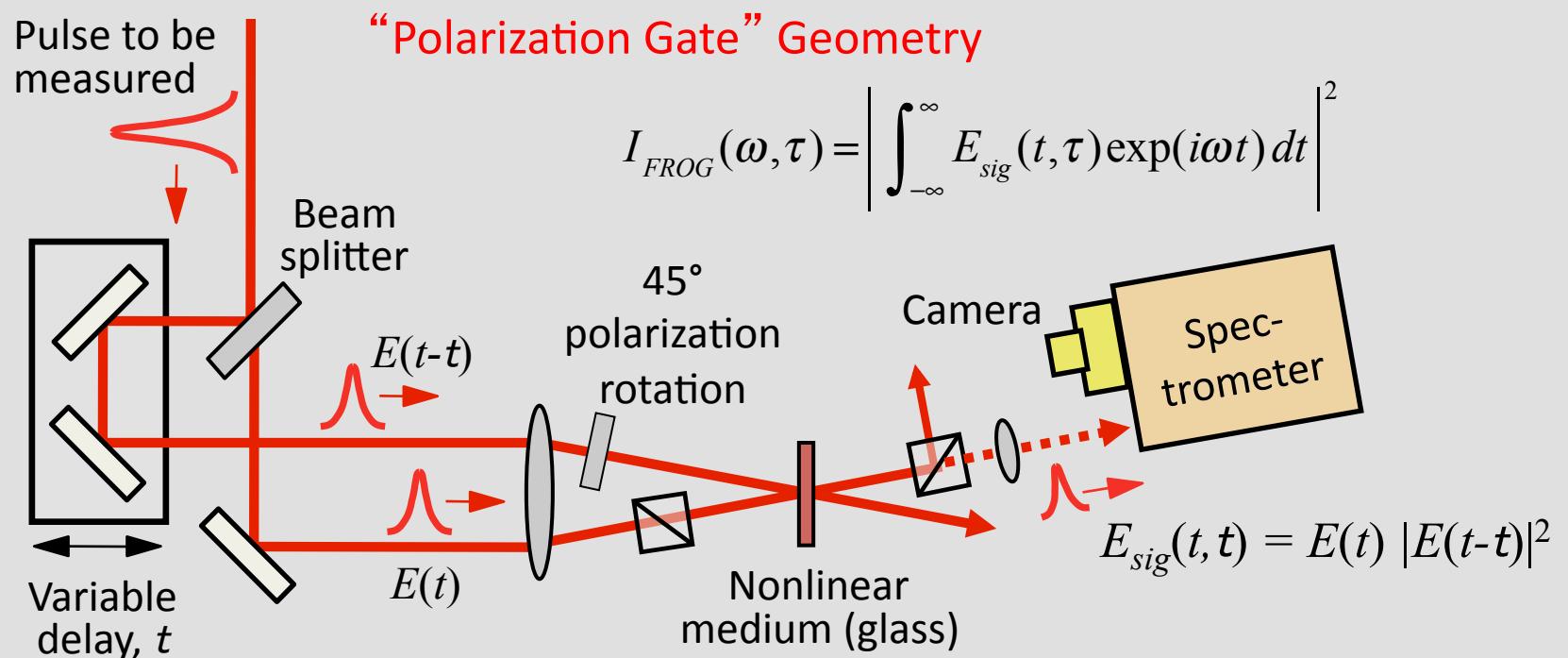
$$N = 4 \times 10^{22} \text{ cm}^{-3}, \quad d = 3 \times 10^{-8} \text{ cm}, \quad e = 4.8 \times 10^{-10} \text{ esu}$$

$$\omega_0 = 7 \times 10^{15} \text{ rad/s}, \quad m = 9.1 \times 10^{-28} \text{ g}$$

$$\Rightarrow \underline{\chi^{(3)} \sim 2 \times 10^{-14} \text{ esu}}$$

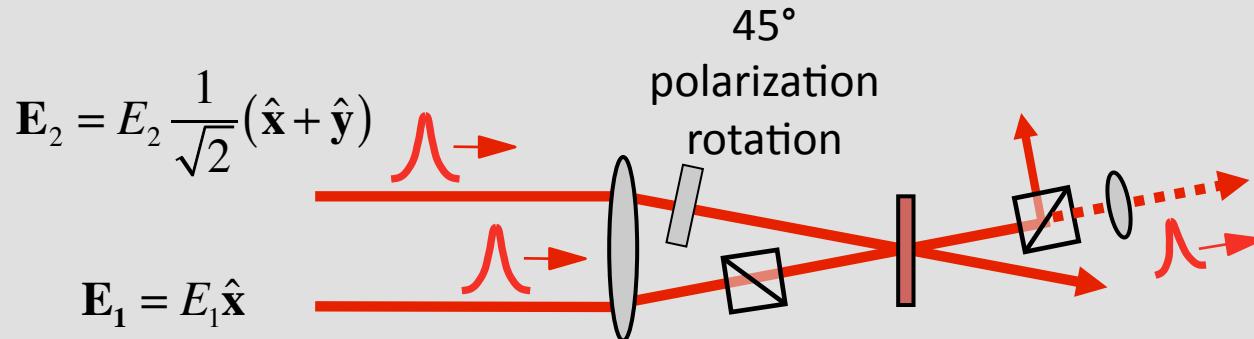
# Polarization gating

- Tensor nature of  $\chi^{(3)}$  allows rotation of polarization by a gating beam



This geometry can also be used as an optical shutter to image fast events.

# NL polarization in the interaction



$$\mathbf{P} = A(\mathbf{E} \cdot \mathbf{E}^*)\mathbf{E} + \frac{1}{2}B(\mathbf{E} \cdot \mathbf{E})\mathbf{E}^*$$

We detect the y component of the output, in the direction of probe:

$$P_y = A(\mathbf{E} \cdot \mathbf{E}^*)E_{2y} + \frac{1}{2}B(\mathbf{E} \cdot \mathbf{E})E_{2y}^*$$

where

$$\mathbf{E} = \mathbf{E}_1 e^{i\mathbf{k}_1 \cdot \mathbf{r}} + \mathbf{E}_2 e^{i\mathbf{k}_2 \cdot \mathbf{r}} = \hat{\mathbf{x}} \left( E_1 e^{i\mathbf{k}_1 \cdot \mathbf{r}} + \frac{1}{\sqrt{2}} E_2 e^{i\mathbf{k}_2 \cdot \mathbf{r}} \right) + \hat{\mathbf{y}} \frac{1}{\sqrt{2}} E_2 e^{i\mathbf{k}_2 \cdot \mathbf{r}}$$

$$\begin{aligned} \mathbf{E} \cdot \mathbf{E}^* &= \left( E_1 e^{i\mathbf{k}_1 \cdot \mathbf{r}} + \frac{1}{\sqrt{2}} E_2 e^{i\mathbf{k}_2 \cdot \mathbf{r}} \right) \left( E_1^* e^{-i\mathbf{k}_1 \cdot \mathbf{r}} + \frac{1}{\sqrt{2}} E_2^* e^{-i\mathbf{k}_2 \cdot \mathbf{r}} \right) + \frac{1}{2} |E_2|^2 \\ &= |E_1|^2 + \frac{1}{2} |E_2|^2 + \frac{1}{\sqrt{2}} E_1 E_2^* e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}} + \frac{1}{\sqrt{2}} E_2 E_1^* e^{-i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}} + \frac{1}{2} |E_2|^2 \end{aligned}$$

$$\mathbf{E} \cdot \mathbf{E} = \left( E_1 e^{i\mathbf{k}_1 \cdot \mathbf{r}} + \frac{1}{\sqrt{2}} E_2 e^{i\mathbf{k}_2 \cdot \mathbf{r}} \right)^2 + \frac{1}{2} E_2^2 = E_1^2 e^{2i\mathbf{k}_1 \cdot \mathbf{r}} + \frac{1}{2} E_2^2 e^{2i\mathbf{k}_2 \cdot \mathbf{r}} + \frac{2}{\sqrt{2}} E_1 E_2 e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}} + \frac{1}{2} E_2^2$$

# Pick output direction and polarization

- Output polarization is in y direction

$$P_y e^{+i\mathbf{k}_1 \cdot \mathbf{r}} = A(\mathbf{E} \cdot \mathbf{E}^*) \mathbf{E}_y + \frac{1}{2} B(\mathbf{E} \cdot \mathbf{E}) \mathbf{E}_y = A(\mathbf{E} \cdot \mathbf{E}^*) \frac{1}{\sqrt{2}} \mathbf{E}_2 e^{+i\mathbf{k}_2 \cdot \mathbf{r}} + \frac{1}{2} B(\mathbf{E} \cdot \mathbf{E}) \frac{1}{\sqrt{2}} \mathbf{E}_2^* e^{-i\mathbf{k}_2 \cdot \mathbf{r}}$$

- Look for output in the  $\mathbf{k}_1$  direction: find combo with  $\mathbf{E}_2$  and  $\mathbf{E}_2^*$

$$\mathbf{E} \cdot \mathbf{E}^* = |E_1|^2 + \frac{1}{2}|E_2|^2 + \frac{1}{\sqrt{2}} E_1 E_2^* e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}} + \frac{1}{\sqrt{2}} E_2 E_1^* e^{-i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}} + \frac{1}{2}|E_2|^2$$

$$\mathbf{E} \cdot \mathbf{E} = E_1^2 e^{2i\mathbf{k}_1 \cdot \mathbf{r}} + \frac{1}{2} E_2^2 e^{2i\mathbf{k}_2 \cdot \mathbf{r}} + \frac{2}{\sqrt{2}} E_1 E_2 e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}} + \frac{1}{2} E_2^2$$

$$\begin{aligned} P_y &= A \frac{1}{\sqrt{2}} E_1 E_2^* \frac{1}{\sqrt{2}} E_2 + \frac{1}{2} B \frac{2}{\sqrt{2}} E_1 E_2 \frac{1}{\sqrt{2}} E_2^* \\ &= A \frac{1}{2} E_1 |E_2|^2 + \frac{1}{2} B E_1 |E_2|^2 = \frac{1}{2}(A+B) E_1 |E_2|^2 \end{aligned}$$

- $E_1$  gated by a real quantity, so the phase of  $P_y$  is that of  $E_1$

# Circular polarization

- Any polarization state can be written as a linear combination of circular basis vectors

$$\mathbf{E} = E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-$$

- Circular polarization basis vectors are a linear combination of x, y components with 90 deg phase shift

$$\hat{\sigma}_+ = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i \hat{\mathbf{y}}) \quad \hat{\sigma}_- = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i \hat{\mathbf{y}})$$

- Properties:

$$\hat{\sigma}_+^* = \hat{\sigma}_- \quad \text{General case} \quad \hat{\sigma}_\pm^* = \hat{\sigma}_\mp$$

- To calculate intensity from vector field:  $I \propto |\mathbf{E}|^2 \equiv \mathbf{E} \cdot \mathbf{E}^*$
- This is how we calculate normalization of unit vectors:

$$|\hat{\sigma}_+|^2 = \hat{\sigma}_+ \cdot \hat{\sigma}_+^* = \hat{\sigma}_+ \cdot \hat{\sigma}_- = 1$$

- And orthogonality:

$$\hat{\sigma}_+ \cdot \hat{\sigma}_-^* = \hat{\sigma}_+ \cdot \hat{\sigma}_+ = 0$$

# NL response to field in circular basis

$$\mathbf{P}^{NL} = A(\mathbf{E} \cdot \mathbf{E}^*)\mathbf{E} + \frac{1}{2}B(\mathbf{E} \cdot \mathbf{E})\mathbf{E}^* \quad \mathbf{E} = E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-$$

$$\mathbf{E} \cdot \mathbf{E}^* = |E_+|^2 + |E_-|^2$$

$$\begin{aligned}\mathbf{E} \cdot \mathbf{E} &= (E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-) \cdot (E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-) \\ &= E_+^2 \hat{\sigma}_+ \cdot \hat{\sigma}_+ + E_-^2 \hat{\sigma}_- \cdot \hat{\sigma}_- + 2E_+ E_- \hat{\sigma}_+ \cdot \hat{\sigma}_- \\ &= 0 + 0 + 2E_+ E_-\end{aligned}$$

$$\begin{aligned}\mathbf{P}^{NL} &= A(|E_+|^2 + |E_-|^2)\mathbf{E} + \frac{1}{2}B(2E_+ E_-)\mathbf{E}^* \\ &= A(|E_+|^2 + |E_-|^2)(E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-) + B E_+ E_- (E_+^* \hat{\sigma}_+^* + E_-^* \hat{\sigma}_-^*)\end{aligned}$$

# Separate NL polarization into + and - components

$$\begin{aligned}\mathbf{P}^{NL} &= A \left( |E_+|^2 + |E_-|^2 \right) (E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-) + B E_+ E_- (E_+^* \hat{\sigma}_+^* + E_-^* \hat{\sigma}_-^*) \\ &= A \left( |E_+|^2 + |E_-|^2 \right) (E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_-) + B E_+ E_- (E_+^* \hat{\sigma}_- + E_-^* \hat{\sigma}_+)\end{aligned}$$

$$P_+ = A \left( |E_+|^2 + |E_-|^2 \right) E_+ + B E_+ \mathbf{E}_- E_-^*$$

$$= A \left( |E_+|^2 + |E_-|^2 \right) E_+ + B |\mathbf{E}_-|^2 E_+$$

$$= \left( A |E_+|^2 + (A + B) |E_-|^2 \right) E_+$$

$$P_+ = \chi_+ E_+$$

$$\boxed{\chi_+ = A |E_+|^2 + (A + B) |E_-|^2}$$

$$P_- = \chi_- E_-$$

$$\boxed{\chi_- = A |E_-|^2 + (A + B) |E_+|^2}$$

# Tensor response leads to induced optical activity

- “Optically active” materials have a refractive index that is different for R and L circular basis
  - Typically chiral molecules respond this way
- Here, the NL response breaks down into R and L (+ and -) symmetry:

$$n = \sqrt{1 + \chi^{(1)} + 3\chi^{(3)}|E|^2} = \sqrt{n_0^2 + \chi^{NL}}$$

$$n_{\pm} = \sqrt{n_0^2 + \chi_{\pm}} \approx n_0 \left( 1 + \frac{1}{2n_0^2} \chi_{\pm} \right)$$

$$\boxed{\chi_+ = A|E_+|^2 + (A+B)|E_-|^2}$$

$$\boxed{\chi_- = A|E_-|^2 + (A+B)|E_+|^2}$$

# NL propagation: circular input

- If the input is pure circular polarization:

$$\mathbf{E}_{in} = E_+ \hat{\sigma}_+ \quad \mathbf{E}_{out} = E_+ e^{i k z} \hat{\sigma}_+ = E_+ \exp[i k_0 n_+ z] \hat{\sigma}_+$$

$$n_+ = n_0 + \frac{1}{2n_0} \chi_+ = n_0 + \frac{1}{2n_0} \left( A |E_+|^2 + (A+B) |E_-|^2 \right)$$

In this case,  $E_- = 0$

$$n_+ = n_0 + \frac{1}{2n_0} A |E_+|^2$$

$$\mathbf{E}_{out} = E_+ \exp \left[ i k_0 \left( n_0 + \frac{1}{2n_0} A |E_+|^2 \right) z \right] \hat{\sigma}_+$$

$$= E_+ e^{i k z} \exp \left[ i k_0 \frac{A}{2n_0} |E_+|^2 z \right] \hat{\sigma}_+$$

Here we have a NL phase shift, but no change in polarization

# NL propagation, general input

- Treat polarization as a vector  $\mathbf{E}_{in} = E_+ \hat{\sigma}_+ + E_- \hat{\sigma}_- \Rightarrow \begin{pmatrix} E_+ \\ E_- \end{pmatrix}$
- After propagation,  $\mathbf{E}_{out} = \begin{pmatrix} E_+ e^{i k_0 n_+ L} \\ E_- e^{i k_0 n_- L} \end{pmatrix} = e^{i k_0 n_- L} \begin{pmatrix} E_+ e^{i k_0 (n_+ - n_-) L} \\ E_- \end{pmatrix}$

$$\begin{aligned}
\Delta n = n_+ - n_- &= \frac{1}{2n_0} \left( A|E_+|^2 + (A+B)|E_-|^2 \right) - \frac{1}{2n_0} \left( A|E_-|^2 + (A+B)|E_+|^2 \right) \\
&= \frac{A}{2n_0} \left( |E_+|^2 - |E_-|^2 \right) + \frac{A+B}{2n_0} \left( |E_-|^2 - |E_+|^2 \right) \\
&= \left( \frac{A}{2n_0} - \frac{A+B}{2n_0} \right) \left( |E_+|^2 - |E_-|^2 \right) \\
&= \frac{B}{2n_0} \left( |E_-|^2 - |E_+|^2 \right)
\end{aligned}$$

For linear polarization input, we have a NL phase shift that is the same for both components, so no change in polarization

# NL ellipse rotation

- With elliptical input (neither circular or linear)

$$\mathbf{E}_{out} = e^{i k_0 n_- L} \begin{pmatrix} E_+ e^{i k_0 \Delta n L} \\ E_- \end{pmatrix} = e^{i k_0 (n_- + \Delta n/2) L} \begin{pmatrix} E_+ e^{i k_0 \Delta n L/2} \\ E_- e^{-i k_0 \Delta n L/2} \end{pmatrix} \quad n_- + \frac{1}{2} \Delta n = \frac{1}{2} (n_+ + n_-)$$

- This is what happens when the coordinates were rotated. The ellipticity remains the same.

$$\begin{aligned} \mathbf{E} &= E_+ \hat{\sigma}_+ e^{i\theta} + E_- \hat{\sigma}_- e^{-i\theta} \\ &= E_+ \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} + i \hat{\mathbf{y}})(\cos\theta + i \sin\theta) + E_- \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} - i \hat{\mathbf{y}})(\cos\theta - i \sin\theta) \\ &= E_+ \frac{1}{\sqrt{2}} \{ (\hat{\mathbf{x}} \cos\theta - \hat{\mathbf{y}} \sin\theta) + i (\hat{\mathbf{x}} \sin\theta + \hat{\mathbf{y}} \cos\theta) \} + \\ &\quad E_- \frac{1}{\sqrt{2}} \{ (\hat{\mathbf{x}} \cos\theta - \hat{\mathbf{y}} \sin\theta) - i (\hat{\mathbf{x}} \sin\theta + \hat{\mathbf{y}} \cos\theta) \} \\ &= E_+ \frac{1}{\sqrt{2}} (\hat{\mathbf{x}}' + i \hat{\mathbf{y}}') + E_- \frac{1}{\sqrt{2}} (\hat{\mathbf{x}}' - i \hat{\mathbf{y}}') = E_+ \hat{\sigma}'_+ + E_- \hat{\sigma}'_- \end{aligned}$$