**MATH 332 - Linear Algebra** August 3, 2009

**Homework 9, Summer 2009** **Due:** August 5, 2009

## Least Squares - Orthogonal Diagonalization - Spectral Decomposition - Singular Value Decomposition

1. Given,

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ -5 & -1 \\ 1 & 2 \end{bmatrix}.$$

Determine an orthonormal basis for the column space of $\mathbf{A}$.

2. Given the linear system of equations,

$$x_1 + x_3 = 2$$

$$x_1 + x_2 = 4$$

   (a) Determine the least-squares solution to the linear system.

   (b) Determine the least-squares error associated with the linear system.

   (c) Graph the linear system, the least-squares solution, and the least-squares error in $\mathbb{R}^2$.

3. Given,

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}.$$

   (a) Show that the columns of $\mathbf{A}$ are linearly independent.

   (b) Determine the $\mathbf{QR}$ factorization of $\mathbf{A}$.

   (c) Using this factorization calculate the unique least-squares solution $\hat{\mathbf{x}} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}$.[1]

4. Recall the Pauli Spin Matrix from a previous homework,

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

   (a) Show that $\sigma_y$ is self-adjoint.[2]

   (b) Find the orthogonal diagonalization of $\sigma_y$.[3]

   (c) Show that $\sigma_y = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^H + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^H$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the normalized eigenvectors from part (b).[4]

5. Given,

$$\mathbf{A} = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}.$$

Find a singular value decomposition of $\mathbf{A}$.

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[1] See theorem 6.5.15 on page 414.

[2] If a matrix is equal to its transpose then we say that the matrix is symmetric. If the matrix has complex numbers then we have a more general definition. For $\mathbf{A} \in \mathbb{C}^{n\times n}$ we say that $\mathbf{A}$ is self-adjoint if $\mathbf{A}^H = \bar{\mathbf{A}}^T = \mathbf{A}$. That is, a matrix is self-adjoint if it is equal to its own complex-conjugate transpose. Self-adjoint matrices are the analouge of symmetric matrices for the complex number field. Also, notice that this definition recovers the definition of symmetric if the matrix has only real entries.

[3] You should find eigenvectors with complex entries. If you use the standard definition of inner-product then you will get zero length, which is sensible since part of the direction of these vectors is into the complex number system. However, this will lead you to a division by zero when trying to normalise the eigenvector. In the case where vectors have complex entries the inner-product is generalised to $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^H \mathbf{y} = \bar{\mathbf{x}}^T \mathbf{y}$. Notice, again that the standard definition of inner-product is recovered when the vectors are real.

[4] This is called the spectral decomposition of a self-adjoint matrix. It is interesting to note that $\mathbf{x}_i^H \mathbf{x}_i = 1$, which implies that the 'matrix' has the same structure as the unitary matrices of chapter 6. Since, $\mathbf{x}_i \mathbf{x}_i^H$ is not the identity matrix we have that $\mathbf{x}_i \mathbf{x}_i^H$ has the same structure as unitary matrices from problem 3 in homework 8. Consequently, we conclude that the self-adjoint matrix has been decomposed into projection matrices, which project an arbitrary vector into eigen-subspaces of the original matrix.

5. A basis for the column space of A is $B_{ColA} = \{\vec{a}_1, \vec{a}_2\}$. Since $\vec{a}_1^T\vec{a}_2 \neq 0$ $B_{ColA}$ is not an orthogonal basis.

To form an orthogonal basis for the plane spanned by $\vec{a}_1, \vec{a}_2$ in $\mathbb{R}^3$ we use Gram-Schmidt.

Thus

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \vec{a}_1$$

$$\vec{v}_2 = \vec{a}_2 - \frac{\vec{a}_2^T \vec{v}_1}{\vec{v}_1^T \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{[4 \ -1 \ 2]\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}}{[2 \ -5 \ 1]\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{30}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.5 \\ 1.5 \end{bmatrix}$$

Note $\vec{v}_1^T\vec{v}_2 = 6 - 7.5 + 1.5 = 0$

Thus $B_\perp = \{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for ColA.
To form the orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ we normalize $\vec{v}_1, \vec{v}_2$.

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{30}}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \frac{\sqrt{2}}{\sqrt{27}}\begin{bmatrix} 3 \\ 1.5 \\ 1.5 \end{bmatrix}$$

2. (1) $x_1 + x_2 = 2$ $\Rightarrow$ $A\vec{x} = \vec{b}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(2) $x_1 + x_2 = 1$

(a)

Clearly these lines never intersect. To solve the least squares problem for $A\vec{x} = \vec{b}$ we solve,

$$A^T A \vec{x} = A^T \vec{b}$$

where

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

and

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Which gives the augmented matrix,

$$\left[\begin{array}{cc|c} 2 & 2 & 6 \\ 2 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 + x_2 = 3$$ defines an $\infty$-amount of solutions.

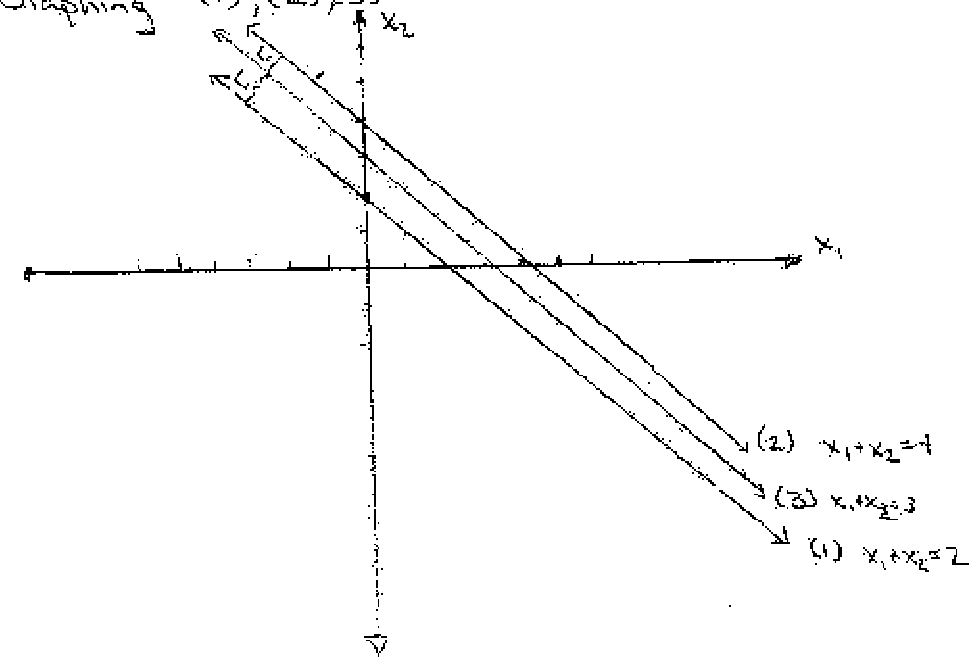
In vector form this is: $\vec{x} = \begin{bmatrix} 3 - x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (3)

(b). The least squares error is given by

$$\|\vec{b} - \hat{b}\| = \left\| \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 - x_2 \\ x_2 \end{bmatrix}}_{A\hat{x}} \right\| = \left\| \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 - x_2 + x_2 \\ 3 - x_2 + x_2 \end{bmatrix} \right\| =$$

$$= \left\| \begin{bmatrix} 2-3 \\ 4-3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \sqrt{2}$$

c. Graphing (1), (2), (3)




where $L=\sqrt{2}$. That is the distance between (1),(3) and (2),(3) is $\sqrt{2}$.

3. Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

a. To show that the columns are linearly independent you can Row Reduce $[A \mid 0]$ to show that only the trivial solution exists.

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$$

It is also pretty clean by looking at them that there is no constant $c \in \mathbb{R}$ s.t. $\bar{a}_1 = c\,\bar{a}_2$.

b. Using $\bar{a}_1, \bar{a}_2$ apply G.S.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \frac{6+8+1}{4+4+1} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

Normalizing gives.

$$\vec{q}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{q}_2 = \frac{1}{\|\vec{v}_2\|} \vec{v}_2 = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

Which implies $Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}$ and

$$R = Q^T A = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix} \Rightarrow R^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & -5 \\ 0 & 3 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & -5 \\ 0 & 3 \end{bmatrix}$$

c. Since the columns of $A$ are linearly independent a unique solution to $A^TA\bar{x} = A^T\bar{b}$ Exists. Theorem 6.5.15 gives this unique solution in terms of the QR factorization of $A$ as

$$\bar{x} = \underset{2\times2}{\frac{1}{3}\begin{bmatrix} 1 & -5 \\ 0 & 3 \end{bmatrix}} \underset{2\times3}{\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}} = \underset{2\times1}{\begin{bmatrix} 4 \\ -1 \end{bmatrix}}$$

4. Let $\hat{S}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

a. $\hat{S}_y^H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \hat{S}_y \Rightarrow \hat{S}_y$ is self adjoint.

b. $\det(\hat{S}_y^H - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\underline{\lambda_1 = 1}$

$[\hat{S}_y - 1 I \mid 0] = \left[\begin{array}{cc|c} -1 & -i & 0 \\ i & -1 & 0 \end{array}\right] \Rightarrow -x_1 - ix_2 = 0$

$\Rightarrow x_1 = -ix_2 \qquad \vec{x} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} \Rightarrow$

$x_2$ free

$\Rightarrow \vec{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \vec{v}_1 = \dfrac{1}{\|\vec{x}_1\|}\vec{x}_1 = \dfrac{1}{\sqrt{x^T x}}\begin{bmatrix} -i \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$ (normalized Eigenvector)

$\underline{\lambda_2 = -1}$

$[\hat{S}_y + I \mid 0] = \left[\begin{array}{cc|c} 1 & -i & 0 \\ i & 1 & 0 \end{array}\right] \Rightarrow x_1 = ix_2 \qquad \vec{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \vec{v}_2 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$x_2$ free

Thus,

$$A = \underset{P}{\begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \underset{P^H}{\begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}},$$

$$PP^H = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 + 1/2 & -i/2 + i/2 \\ i/2 - i/2 & 1/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. $A = \lambda_1 \vec{v}_1 \vec{v}_1^H + \lambda_2 \vec{v}_2 \vec{v}_2^H =$

$$= 1 \cdot \begin{bmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} +i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} - 1 \cdot \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} =$$

$$= \begin{bmatrix} +1/2 & -i/2 \\ +i/2 & 1/2 \end{bmatrix} - \begin{bmatrix} +1/2 & i/2 \\ -i/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

5

If $A = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}$ then $A^T A = \begin{bmatrix} 7 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}$

$\det(A - \lambda I) = \det \begin{bmatrix} 74-\lambda & 32 \\ 32 & 26-\lambda \end{bmatrix} = (26-\lambda)(74-\lambda) - 32^2 =$

$= \lambda^2 - 100\lambda + 900 = (\lambda - 10)(\lambda - 90) = 0$

$\Rightarrow \lambda_1 = 90$ (*) Note I have placed them in descending order for later.
$\lambda_2 = 10$

$\lambda_1 = 90$:

$[A - \lambda I | 0] = \left[\begin{array}{cc|c} 74-90 & 32 & 0 \\ 32 & 26-90 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -16 & 32 & 0 \\ 32 & -64 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} +1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ $x_1 = 2x_2$, $x_2$ is free

$\Rightarrow \vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2 \underset{\text{normalizing}}{\Rightarrow} \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 10$

$\left[\begin{array}{cc|c} 64 & 32 & 0 \\ 32 & 16 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -2x_1 = x_2$, $x_1$ free $\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Rightarrow \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, Thus $\sigma_1 = \sqrt{90}$, $\sigma_2 = \sqrt{10}$, $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

implies that

$$A_{3\times 2} = U_{3\times 3} \Sigma_{3\times 2} V^T_{2\times 2}$$

where $\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$ $\quad V = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$

To find $U_{3\times 3}$,

$$\vec{u}_1 = \underbrace{\frac{1}{\|A\vec{v}_1\|}}_{\xrightarrow{\text{note!}}} A\vec{v}_1 = \frac{1}{\sigma_1}\begin{bmatrix} 7\cdot 2/\sqrt{5} + 1/\sqrt{5} \\ 0 \\ 10/\sqrt{5} + 5/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{90}}\begin{bmatrix} 15/\sqrt{5} \\ 0 \\ 15/\sqrt{5} \end{bmatrix}$$

$$\vec{u}_2 = \underbrace{\frac{1}{\|A\vec{v}_2\|}}_{\xrightarrow{\text{note!}}} A\vec{v}_2 = \frac{1}{\sigma_2}\begin{bmatrix} 7/\sqrt{5} - 2/\sqrt{5} \\ 0 \\ 5/\sqrt{5} - 10/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 5/\sqrt{5} \\ 0 \\ -5/\sqrt{5} \end{bmatrix}$$

To find $\vec{u}_3$ determine the normalized $x \in \mathbb{R}^3$ s.t.
$\vec{u}_1^T\vec{x} = 0$, $\vec{u}_2^T\vec{x} = 0$

$$\vec{u}_1^T\vec{x} = \left(\frac{15}{\sqrt{5}}x_1 + \frac{15}{\sqrt{5}}x_2\right)\frac{1}{\sqrt{90}} = 0 \Rightarrow \left[\begin{array}{ccc|c} 15 & 0 & 15 & 0 \\ 5 & 0 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 30 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \end{array}\right]$$

$$\vec{u}_2^T\vec{x} = \left(\frac{5}{\sqrt{5}}x_1 + \frac{-5}{\sqrt{5}}x_2\right)\frac{1}{\sqrt{10}} = 0$$

$\Rightarrow$ $x_1 = 0$ $\quad x_2 =$ free
$x_3 = 0$

and,

$$\vec{u}_3 = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{choose } \vec{x}_2 \text{ so that } \|\vec{u}_3\| = 1$$

Then $$U = \begin{bmatrix} 15/\sqrt{450} & 5/\sqrt{50} & 0 \\ 0 & 0 & 1 \\ 15/\sqrt{450} & -5/\sqrt{50} & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 15/\sqrt{450} & 5/\sqrt{50} & 0 \\ 0 & 0 & 1 \\ 15/\sqrt{450} & -5/\sqrt{50} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$