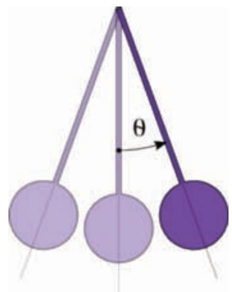
MATH 225 - Differential Equations May 27 , 2008
Homework 5, Field 2008 **Due Date**: May 29, 2008

SYSTEMS OF FIRST ORDER ODE S - LINEAR EQUATIONS - REAL EIGENVALUES - PHASE PORTRAITS

1. Suppose we have massless rod of length L, which is fixed to frictionlessly rotate about an endpoint with ridged mass m fixed at the other endpoint. Choosing the coordinate system depicted in the following diagram,



allows one to derive the following second-order differential equation,[1]

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{mg}{L}\sin(\theta) = f(t) \tag{1}$$

where the dependent variable θ measures the radial displacement from equilibrium as a function of time. [2]

(a) Using the substitution $\frac{d\theta}{dt} = \omega$ derive a system of first order ordinary differential equations, which models the displacement of the pendulum, from equilibrium, as a function of time.

(b) Classify the *type, order, and linearity* of the previous system of differential equations.

(c) Assuming that $\theta \ll 1$ and derive a linear system of differential equations, which models the motion of the pendulum.

2. Given,

$$\frac{dx}{dt} = x \tag{2}$$

$$\frac{dy}{dt} = -y \tag{3}$$

(a) Using eigenvalues and eigenvectors, find the general solution of this system.

(b) Graph, by hand, the phase portrait associated with the system and using HPGSYSTEMSOLVER check your work.

(c) Find and classify any equilibrium solutions.

(d) Noticing that the system is decoupled, solve each ODE by the methods of chapter 1 and show that this yields the same solution you found in part (a).

3. Given that $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$.

(a) Find the general solution of this system.

(b) Using HPGSYSTEMSOLVER plot the phase portrait and classify the equilibrium solution.

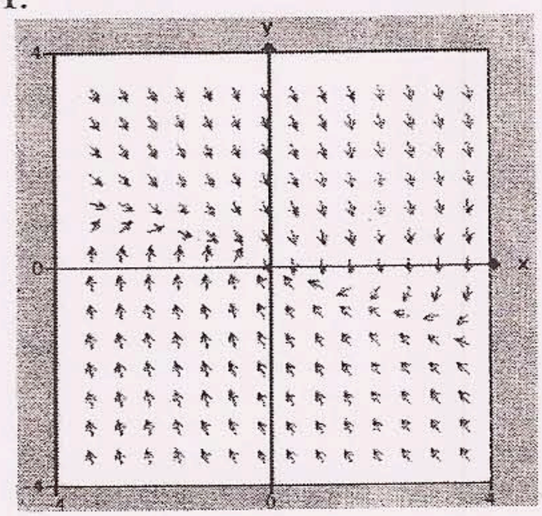
[1] See chapter 5 of text for details.
[2] Similar to the ideal mass spring system from before we have $b \equiv$ coefficient of kinetic friction and $f(t) \equiv$ applied force.

4. Given that $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$. Describe how the phase portrait of the system changes as α 0^+ and α 0^-.

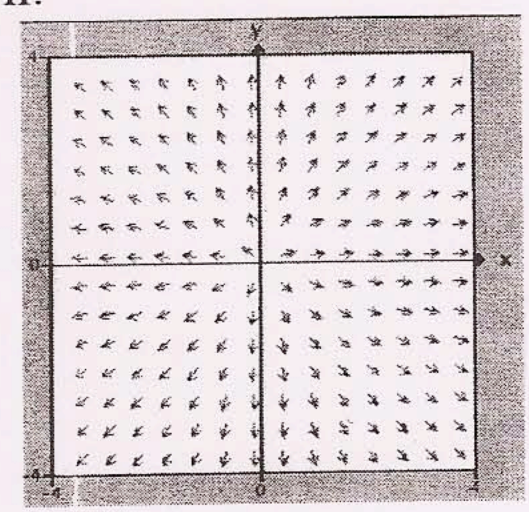
5. Match the following systems with the corresponding direction fields.

(a) $\dfrac{dx}{dt} = y$ $\dfrac{dy}{dt} = -x - 3y$	(b) $\dfrac{dx}{dt} = x$ $\dfrac{dy}{dt} = y$	(c) $\dfrac{dx}{dt} = \sin(y)$ $\dfrac{dy}{dt} = \cos(x)$	(d) $\dfrac{dx}{dt} = y$ $\dfrac{dy}{dt} = x$

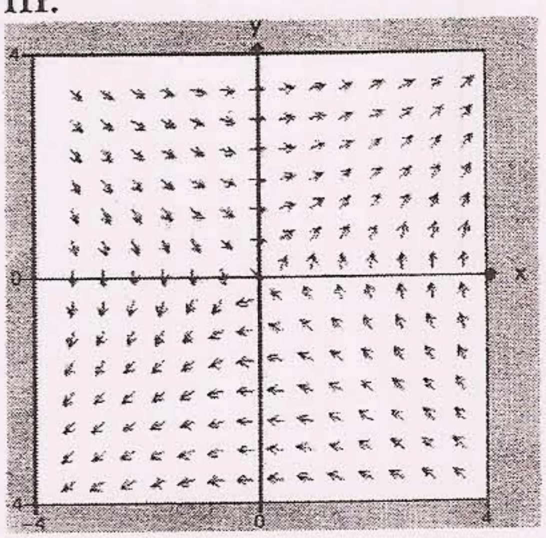
I.



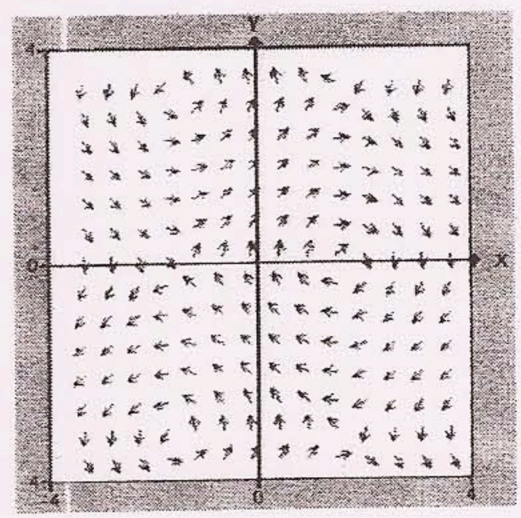
II.



III.



IV.



1)

(1) $m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \frac{mg}{L}\sin(\theta) = f(t)$

a) Let $\frac{d\theta}{dt} = \omega$ then (1) becomes

$$\left.\begin{aligned} \frac{d\omega}{dt} &= -\frac{b}{m}\omega + \frac{g}{L}\sin(\theta) + f(t) \\ \frac{d\theta}{dt} &= \omega \end{aligned}\right\} (2)$$

b) (2) is a system of first order ODE's which is nonlinear in θ.

c) If $\theta \ll 1 \Rightarrow \sin(\theta) \approx \theta$ and (2) becomes

$$\frac{d\omega}{dt} = -\frac{b}{m}\omega - \frac{g}{L}\cancel{\sin(\theta)}^{\theta} + f(t)$$

$$\frac{d\theta}{dt}$$

which is linear in θ.

2) $\frac{dx}{dt} = x \quad , \frac{dy}{dt} = y \iff \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

a) $\det(A - \lambda I) = \cancel{(1-\lambda)}$
$= (1-\lambda)(-1-\lambda) = 0$

$\lambda_1 = -1$
$\lambda_2 = 1$

Case $\lambda_1 = -1$

$(A - \lambda I)\vec{v} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = 0 \Rightarrow \vec{v}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$v_2 \equiv$ anything but $\emptyset$

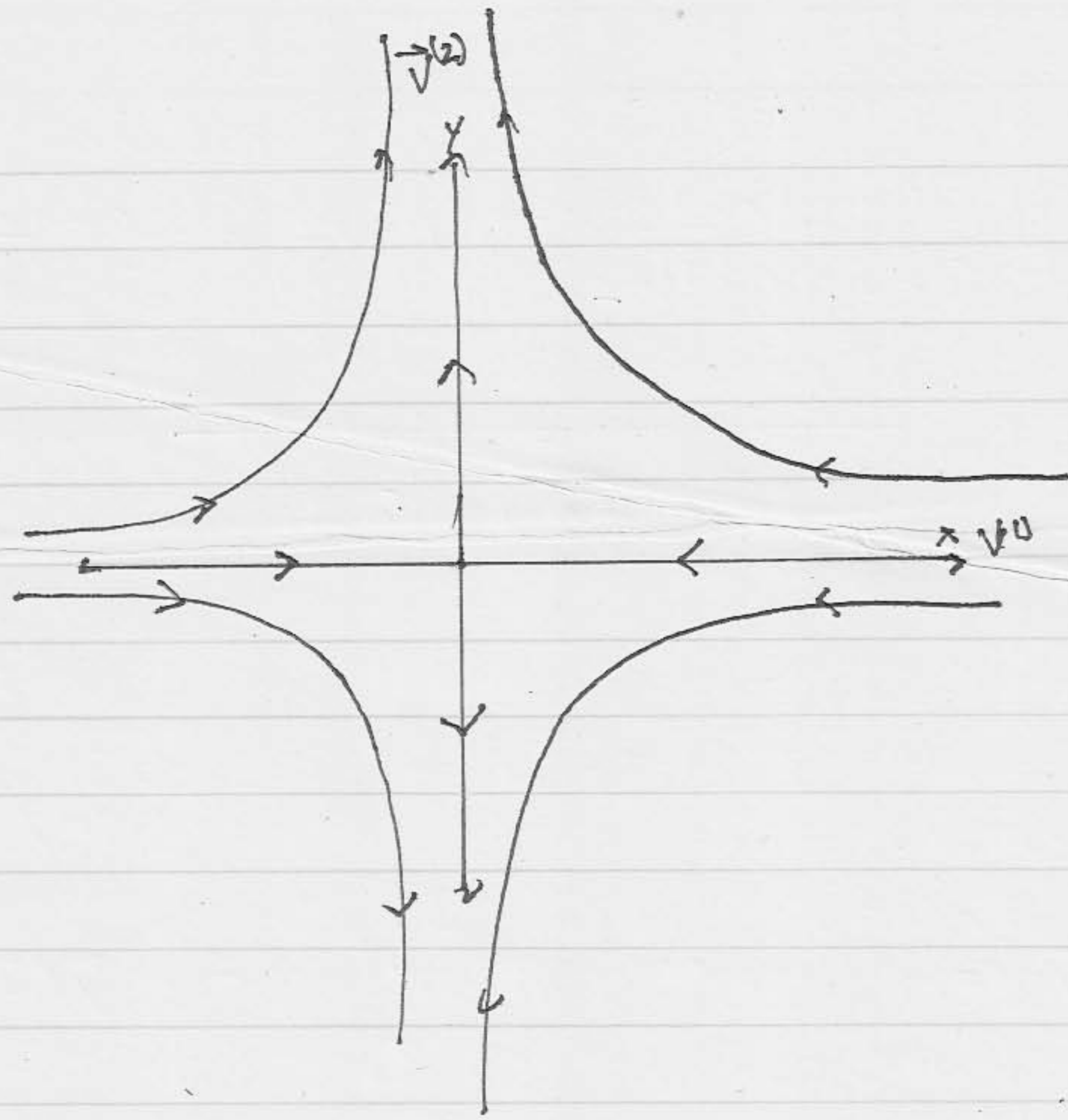
Case $\lambda_2 = 1$

$(A - \lambda I)\vec{v} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

General Soln

$\vec{Y}(t) = k_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-t} + k_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{t} = \begin{bmatrix} k_2 e^{+t} \\ k_1 e^{-t} \end{bmatrix}$

b)



c) $(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a Constant Sol$\underline{n}$ to the problem.

This equilibrium Sol$\underline{n}$ is called a saddle node.

d) $\dfrac{dx}{dt} = x \Rightarrow x(t) = k_1 e^{t}$

$\dfrac{dy}{dt} = -y \Rightarrow y(t) = k_2 e^{-t}$

3) $\frac{d\vec{Y}}{dt} = AY = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

a)

$\det(A-\lambda I) = (-1-\lambda)(-3-\lambda) = 0 \Rightarrow \lambda_1 = -1$
$\lambda_2 = -3$

Case $\lambda_1 = -1$

$(A-\lambda I)\vec{v} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $v_2 = 0$
$v_1 \equiv$ anything but zero.

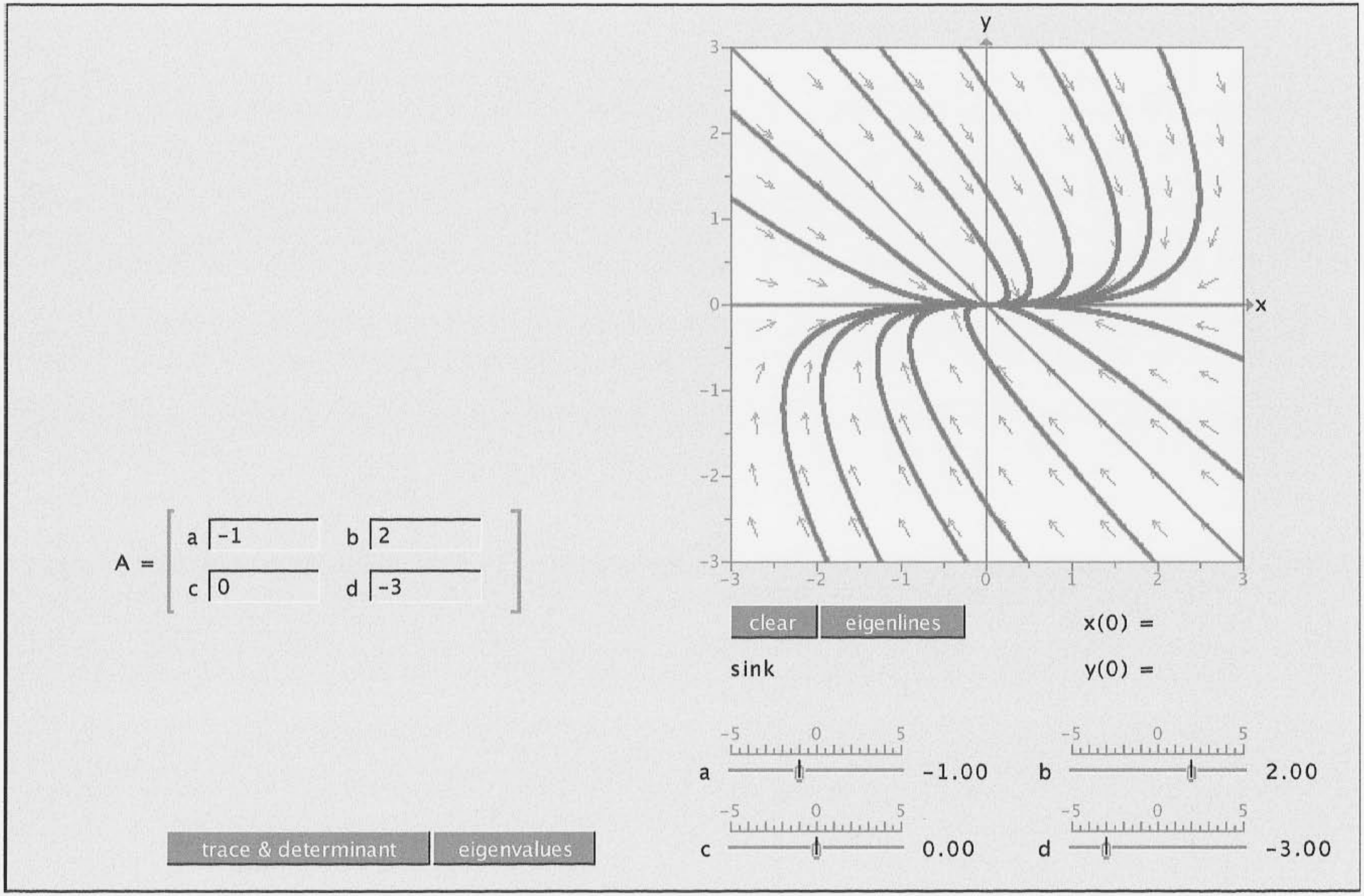
$\vec{V}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Case $\lambda_2 = -3$

$\begin{bmatrix} -1-(-3) & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

General Soln

$\vec{Y}(t) = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + k_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}$


A =
a -1
b 2
c 0
d -3
trace & determinant
eigenvalues
y
x
clear
eigenlines
sink
x(0) =
y(0) =
a -1.00
b 2.00
c 0.00
d -3.00

4) Given,

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \vec{Y}$$

as

$\alpha \to 0^+$ the system has Eigenvalues (Check!)

$\lambda_1 = 1, \ \lambda_2 = \alpha, \quad \alpha > 0$

(0,0) $\Rightarrow$ Real Source

as

$\alpha \to 0^-$ the system has Eigenvalues

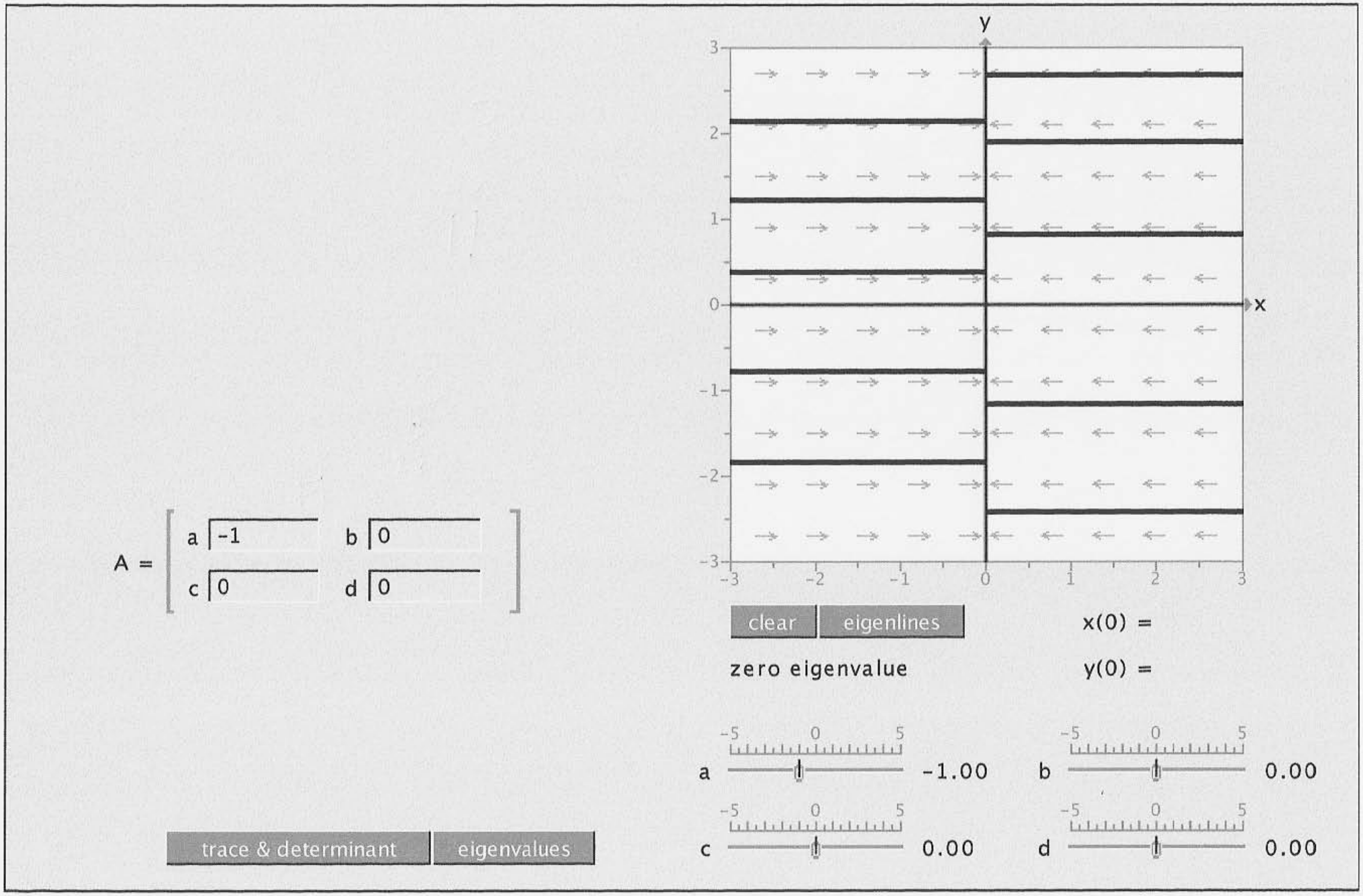
$\lambda_1 = 1, \ \lambda_2 = \alpha, \quad \alpha < 0$

$\Rightarrow$ (0,0) is a saddle.

at $\alpha = 0$ we have the following 2d phase portrait called a comb.

5) (d) $\Leftrightarrow$ III (b) $\Leftrightarrow$ II

(a) $\Leftrightarrow$ I (c) $\Leftrightarrow$ IV


y
x
A =
a -1
b 0
c 0
d 0
trace & determinant
eigenvalues
clear
eigenlines
zero eigenvalue
x(0) =
y(0) =
a -1.00
b 0.00
c 0.00
d 0.00