Reading: Today: G 8.2
Tuesday: G 9.1

Conservation Laws

Charge

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \rightarrow \oint \vec{J}\cdot d\vec{a} = -\frac{\partial Q_{enc}}{\partial t}$$

Energy Cons

$$\vec{\nabla}\cdot\vec{S} = -\frac{\partial u_{em}}{\partial t} - \vec{J}\cdot\vec{E} = -\frac{\partial u_{tot}}{\partial t}$$

Today: Conservation of momentum

$$\vec{F}_{em} = \frac{d\vec{p}_{mech}}{dt} = q\vec{E} + q\vec{v}\times\vec{B}$$

$$= \int \underbrace{\rho\vec{E} + \vec{J}\times\vec{B}}_{\text{Force density on a bit of charge in volume } dV \,\equiv\, \vec{f}}\, dV$$

Force on some charge in a volume V

$$\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$$

$$\vec{F} = \int \vec{f}\, dV$$

$$\left.\begin{aligned} \rho &= \epsilon_0(\vec{\nabla}\cdot\vec{E}) \\ \vec{J} &= \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t} \end{aligned}\right\} \text{M.E.}$$

$$\rho\vec{E} + \vec{J}\times\vec{B} = \left[\epsilon_0(\vec{\nabla}\cdot\vec{E})\right]\vec{E} + \left[\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]\times\vec{B}$$

$$= \epsilon_0\vec{E}(\vec{\nabla}\cdot\vec{E}) + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

Expand the quantity:

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow \rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\vec{E}(\vec{\nabla}\cdot\vec{E}) - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$- \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E}) + \underbrace{\frac{1}{\mu_0}\vec{B}(\vec{\nabla}\cdot\vec{B})}_{\phi \text{ so I'm ok}}$$

$$\Rightarrow \rho\vec{E}+\vec{J}\times\vec{B} = \epsilon_0\vec{E}(\vec{\nabla}\cdot\vec{E}) - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$
$$-\epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E}) + \frac{1}{\mu_0}\vec{B}(\vec{\nabla}\cdot\vec{B})$$

Levi-Civita Density:

$\epsilon_{ijk} = 0$ if, $i=j$, $i=k$, or $j=k$.

$= 1$ if $i=1, j=2, k=3$ or any even permutation of that.

$=-1$ for odd permutations

$$\vec{A}\times\vec{B}\Big|_i = \sum_{j,k=1}^{3}\epsilon_{ijk}A_jB_k \quad ; \quad \vec{A}\cdot\vec{B} = \sum_{i,j=1}^{3}\delta_{ij}A_iB_j$$

$$\vec{\nabla}\times\vec{A}\Big|_i = \sum_{j,k=1}^{3}\epsilon_{ijk}\nabla_jA_k \quad ; \quad \vec{\nabla}\cdot\vec{A} = \sum_{i,j=1}^{3}\delta_{ij}\nabla_iA_j$$

$$\nabla_1 = \frac{\partial}{\partial x},\ \nabla_2 = \frac{\partial}{\partial y},\ \nabla_3 = \frac{\partial}{\partial z}$$

Einstein summation notation:

You sum over repeated indices over all possible indexes {take out Σ symbols}

It turns out that:

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$\vec{A}\times(\vec{B}\times\vec{C})\Big|_i = \epsilon_{ijk}A_j(\epsilon_{klm}B_lC_m) \quad \{\epsilon_{ijk}A_j(\vec{B}\times\vec{C})_k$$

$$= \epsilon_{ijk}\epsilon_{klm}A_jB_lC_m$$

$$= \epsilon_{kij}\epsilon_{klm}A_jB_lC_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m$$

$$= A_jB_iC_j - A_jB_jC_i$$

$$= B_i(A_jC_j) - C_i(A_jB_j)$$

$$= \vec{B}(\vec{A}\cdot\vec{C})\Big|_i - \vec{C}(\vec{A}\cdot\vec{B})\Big|_i$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

What is
$\vec{E}\times(\vec{\nabla}\times\vec{E})$ (no cross products in answer)?

$$E_j \nabla_i E_j - E_j \nabla_j E_i$$

$$\uparrow$$

$$\frac{1}{2}\nabla_i(E_j E_j) = \frac{1}{2}E_j\nabla_i E_j + \frac{1}{2}E_j\nabla_i E_j$$

$$\frac{1}{2}\nabla_i(E^2) - (\vec{E}\cdot\vec{\nabla})E_i$$

$$\Rightarrow \vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\vec{\nabla}(E^2) - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

$$\Rightarrow \rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\vec{E}(\vec{\nabla}\cdot\vec{E}) - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$- \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E}) + \frac{1}{\mu_0}\vec{B}(\vec{\nabla}\cdot\vec{B})$$

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + (\vec{E}\cdot\vec{\nabla})\vec{E}\right]$$

$$+ \frac{1}{\mu_0}\left[\vec{B}(\vec{\nabla}\cdot\vec{B}) + (\vec{B}\cdot\vec{\nabla})\vec{B}\right]$$

$$- \vec{\nabla}\left[\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right]$$

$$- \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

Define <u>Stress Tensor</u>

$$T_{ij} = \epsilon_0 \left[E_i E_j - \tfrac{1}{2}\delta_{ij} E^2\right] + \tfrac{1}{\mu_0}\left[B_i B_j - \tfrac{1}{2}\delta_{ij} B^2\right]$$

$$\begin{bmatrix} \epsilon_0\left(\tfrac{1}{2}E_x^2 - \tfrac{1}{2}E_y^2 - \tfrac{1}{2}E_z^2\right) + \tfrac{1}{\mu_0}\left(\tfrac{1}{2}B_x^2 - \tfrac{1}{2}B_y^2 - \tfrac{1}{2}B_z^2\right) & \epsilon_0 E_x E_y + \tfrac{1}{\mu_0} B_x B_y & \epsilon_0 E_x E_z + \tfrac{1}{\mu_0} B_x B_z \\ \epsilon_0 E_y E_x + \tfrac{1}{\mu_0} B_y B_x & \epsilon_0\left(-\tfrac{1}{2}E_x^2 + \tfrac{1}{2}E_y^2 - \tfrac{1}{2}E_z^2\right) + \tfrac{1}{\mu_0}\left(-\tfrac{1}{2}B_x^2 + \tfrac{1}{2}B_y^2 - \tfrac{1}{2}B_z^2\right) & \epsilon_0 E_y E_z + \tfrac{1}{\mu_0} B_y B_z \\ \epsilon_0 E_z E_x + \tfrac{1}{\mu_0} B_z B_x & \epsilon_0 E_y E_z + \tfrac{1}{\mu_0} B_y B_z & \epsilon_0\left(-\tfrac{1}{2}E_x^2 - \tfrac{1}{2}E_y^2 + \tfrac{1}{2}E_z^2\right) + \tfrac{1}{\mu_0}\left(-\tfrac{1}{2}B_x^2 - \tfrac{1}{2}B_y^2 + \tfrac{1}{2}B_z^2\right) \end{bmatrix}$$

I claim the equation on the last page is actually

$$\underbrace{\rho\vec{E} + \vec{J}\times\vec{B}}_{\substack{\text{time derivative} \\ \text{of mech momentum} \\ \text{density.}}} = \underbrace{\vec{\nabla}\cdot\overleftrightarrow{T}}_{\substack{\text{Div of} \\ \text{Momentum} \\ \text{flow}}} - \frac{\partial}{\partial t}\underbrace{\left(\epsilon_0 \vec{E}\times\vec{B}\right)}_{\text{EM momentum}}$$

$$= \epsilon_0 \mu_0 \vec{S} = \frac{1}{c^2}\vec{S}$$

In integral form

$$\underbrace{\int_V \rho\vec{E} + \vec{J}\times\vec{B}\, dV}_{\substack{\text{Time deriv.} \\ \text{of } \vec{P}_{mech} \text{ in } V.}} = \underbrace{\oint \overleftrightarrow{T}\cdot d\vec{a}}_{\substack{-\text{Flux of} \\ \text{momentum ent-} \\ \text{ering surface.}}} - \frac{\partial}{\partial t}\underbrace{\int_V \epsilon_0 \vec{E}\times\vec{B}\, dV}_{\substack{\text{EM momentum} \\ \text{in V.}}}$$

$$T_{ij}\, da_j$$

Say $d\vec{a} = dx\, dy\, \hat{z}$

$$\Rightarrow \overleftrightarrow{T}\cdot d\vec{a} = T_{iz}\, dx\, dy$$

$$= T_{xz}\, dx\, dy\, \hat{x} + T_{yz}\, dx\, dy\, \hat{y} + T_{zz}\, dx\, dy\, \hat{z}$$