MATH 112 Spring 2012
Exam 3

Name: Solutions
Section:

In order to receive full credit, SHOW ALL SUPPORTING WORK. Enclose your final answers in boxes.

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1. (15 points) Evaluate.

(a) $\int_0^\infty x^2 e^{-x}\,dx = \lim_{t\to\infty} \int_0^t x^2 e^{-x}\,dx = \lim_{t\to\infty}\left[-\frac{x^2}{e^x} - \frac{2x}{e^x} - \frac{2}{e^x}\right]_0^t$

$= \lim_{t\to\infty}\left[-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} - (0-0-2)\right]$

$= 0 - 0 - 0 + 2 = \boxed{2}$

$x^2 \quad + \quad e^{-x}$

$2x \quad - \quad -e^{-x}$

$2 \quad + \quad e^{-x}$

$0 \quad\quad -e^{-x}$

$\lim_{t\to\infty} \frac{-t^2}{e^t} \left(\frac{-\infty}{\infty}\right)$

$\stackrel{L}{=} \lim_{t\to\infty} \frac{-2t}{e^t} \left(\frac{-\infty}{\infty}\right)$

$\stackrel{L}{=} \lim_{t\to\infty} \frac{-2}{e^t} = 0$

$\lim_{t\to\infty} \frac{-2t}{e^t} \left(\frac{-\infty}{\infty}\right)$

$\stackrel{L}{=} \lim_{t\to\infty} \frac{-2}{e^t} = 0$

(b) $\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\,dx = \int \left[(x-2) + \frac{2}{x} + \frac{6}{x^2} + \frac{2}{x-3}\right] dx$

$$\begin{array}{r l} & x - 2 + \frac{-18}{x^3-3x^2} \\ x^3 - 3x^2 \,\big) & x^4 - 5x^3 + 6x^2 - 18 \\ - & x^4 - 3x^3 \\ \hline & -2x^3 + 6x^2 \\ & -2x^3 + 6x^2 \\ \hline & 0 - 18 \end{array}$$

$= \boxed{\frac{x^2}{2} - 2x + 2\ln|x| - 6x^{-1} + 2\ln|x-3| + C}$

$\frac{-18}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$

$-18 = Ax(x-3) + B(x-3) + Cx^2$

$-18 = Ax^2 - 3Ax + Bx - 3B + Cx^2$

$A + C = 0, \quad -3A + B = 0, \quad -3B = -18$

$C = 2 \qquad A = -2 \qquad B = 6$

2. (15 points) Determine the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$.

Use Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1+3}} \cdot \frac{\sqrt{n+3}}{2^n(x-3)^n}\right| = \lim_{n\to\infty} 2|x-3|\sqrt{\frac{n+3}{n+4}}$$

$$= 2|x-3| \overset{\text{conv if}}{<} 1, \quad |x-3| < 1/2 \quad \boxed{R = 1/2}$$

$$-1/2 < x-3 < 1/2$$

$$5/2 < x < 7/2$$

At $x = 5/2$, $\sum_{n=1}^{\infty} \frac{2^n(-1/2)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

1.) $a_{n+1} = \frac{1}{\sqrt{n+1+1}} \overset{?}{\leq} \frac{1}{\sqrt{n+1}} = a_n$

$\sqrt{n+1} \overset{\checkmark}{\leq} \sqrt{n+2}$ on $[1,\infty)$

2.) $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \overset{\checkmark}{=} 0$

C by Leibniz Test for Alt. Series

At $x = 7/2$, $\sum_{n=1}^{\infty} \frac{2^n(\frac{1}{2})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

compare to $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ $p = 1/2 < 1$ D p-series

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1 > 0$$

behave same D by Limit Comparison test

Note: Direct Comparison Test fails.

$$\boxed{I = [5/2, 7/2)}$$

3. (12 points)

(a) Write function $f(x) = \dfrac{1}{10+x}$ as power series.

$$\frac{1}{10+x} = \frac{1}{10\left[1+\frac{x}{10}\right]} = \frac{1}{10}\,\frac{1}{1-\left(-\frac{x}{10}\right)} = \frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{x}{10}\right)^n = \frac{1}{10}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{10^n}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{10^{n+1}}}$$

(b) Use the first three terms of the answer from (a) to estimate $\displaystyle\int_0^1 \frac{1}{10+x}\,dx$. (DO NOT ADD TERMS).

$$\int_0^1 \frac{1}{10+x}\,dx = \int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{10^{n+1}}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{10^{n+1}(n+1)}\Bigg]_0^1$$

$$= \frac{x}{10} - \frac{x2}{10^2\cdot 2} + \frac{x3}{10^3\cdot 3} - \frac{x4}{10^4\cdot 4} + \cdots\Bigg]_0^1 \approx \boxed{\frac{1}{10} - \frac{1}{200} + \frac{1}{300}}$$

(c) Determine the upper bound on the error of the approximation found in (b).

$$|R_2| \le a_3 = \frac{1}{10^4\cdot 4} = \boxed{\frac{1}{40{,}000}}$$

4. (8 points) Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2\sqrt{n^2+1}}$ converges absolutely, converges conditionally, or diverges.

1.) $a_{n+1} = \dfrac{1}{2\sqrt{(n+1)^2+1}} \overset{?}{\le} \dfrac{1}{2\sqrt{n^2+1}} = a_n$

$2\sqrt{n^2+1} \overset{\checkmark}{\le} 2\sqrt{n^2+2n+2}$ on $[1, \infty)$

2) $\displaystyle\lim_{n\to\infty}\frac{1}{2\sqrt{n^2+1}} \overset{\checkmark}{=} 0$

c by Leibniz test for AH-series

Consider $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{2\sqrt{n^2+1}}\right| = \sum_{n=1}^{\infty}\frac{1}{2\sqrt{n^2+1}}$

Compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ D Harmonic

$$\lim_{n\to\infty}\frac{\frac{1}{2\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{2\sqrt{n^2+1}}\cdot\frac{n}{1} = \frac{1}{2} > 0$$

behave same

So, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2\sqrt{n^2+1}}$ not abs. conv.

Therefore $\boxed{\text{conditionally Convergent}}$

5. (32 points) Determine if the following series converge or diverge. State any test used. Find the sum when possible.

(a) $-\frac{2}{5}+\frac{2}{15}-\frac{2}{45}+\cdots$

$r=-\frac{1}{3}$

$|r|=\frac{1}{3}<1$ $\boxed{\text{C by GST}}$

$$S=\frac{-\frac{2}{5}}{1-\left(-\frac{1}{3}\right)}=\frac{-\frac{2}{5}}{\frac{4}{3}}=-\frac{2}{5}\left(\frac{3}{4}\right)=\boxed{-\frac{3}{10}}$$

(b) $\sum_{n=1}^{\infty}\frac{2n}{e^{n^2}}$

$f(x)=\frac{2x}{e^{x^2}}$, cont, pos on $[1,\infty)$

$$f'(x)=\frac{e^{x^2}(2)-2x\,e^{x^2}(2x)}{\left(e^{x^2}\right)^2}\overset{?}{<}0$$

$$=\frac{2e^{x^2}(1-4x)}{\left(e^{x^2}\right)^2}<0 \text{ since } 1-4x<0 \text{ on } [1,\infty)$$

$$\int_1^{\infty}\frac{2x}{e^{x^2}}\,dx=\lim_{t\to\infty}\int_1^t\frac{2x}{e^{x^2}}\,dx \qquad u=x^2,\ du=2x\,dx$$

$$=\lim_{t\to\infty}\int_1^{t^2}\frac{du}{e^u}=\lim_{t\to\infty}-e^{-u}\Big]_1^{t^2}$$

$$=\lim_{t\to\infty}\left[-\cancel{e^{-t^2}}^{\,0}+e^{-1}\right]=\frac{1}{e}$$

$\boxed{\text{C by Int Test}}$

Note: Ratio Test also works

(c) $\sum_{n=1}^{\infty}\left(\frac{2n+1}{n}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{2n+1}{n}\right)^n\right|}=\lim_{n\to\infty}\frac{2n+1}{n}=2>1 \quad \boxed{\text{D by Root Test}}$$

(d) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$

$$\frac{1}{(2n-1)(2n+1)}=\frac{A}{2n-1}+\frac{B}{2n+1}$$

$1=A(2n+1)+B(2n-1)$

$1=2An+A+2Bn-B$

$2A+2B=0$, $A-B=1$

$A+B=0$

$2A=1$

$A=1/2$

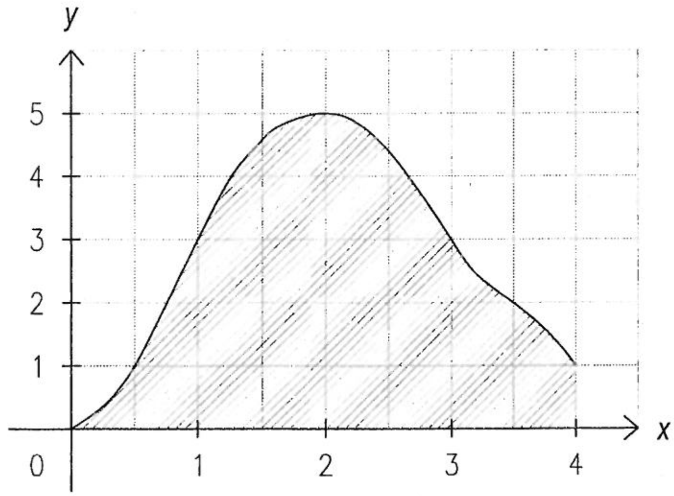
$B=-1/2$

$$S_n=\frac{1}{2}\left[\left(1-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}\right)+\left(\cancel{\tfrac{1}{5}}-\tfrac{1}{7}\right)+\dots+\left(\tfrac{1}{2n-3}-\tfrac{1}{2n-1}\right)+\left(\tfrac{1}{2n-1}-\tfrac{1}{2n+1}\right)\right]$$

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{2}\left[1-\frac{1}{2n+1}\right]$$

$=\boxed{\frac{1}{2}}$ $\boxed{\text{telescoping}}$ $\boxed{\text{C}}$

6. (8 points) Estimate the area under the curve on $[0, 4]$ using Simpson's Rule with $n = 4$.



$h = \frac{4-0}{4} = 1$

$S_4 = \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$

$= \frac{1}{3}[0 + 4(3) + 2(5) + 4(3) + 1]$

$= \frac{1}{3}[12 + 10 + 12 + 1]$

$= \frac{1}{3}[35] = \boxed{\frac{35}{3}}$ units^2

7. (10 points) Consider the sequence with general term $a_n = \dfrac{2 + n^3}{1 + 2n^3}$

(a) Determine if the sequence $\{a_n\}$ converges or diverges. If it converges, determine what value it converges to.

$\lim_{n \to \infty} \frac{2+n^3}{1+2n^3} = \frac{1}{2}$

$\boxed{\{a_n\} \text{ C to } 1/2}$

(b) Determine if the the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. Show all necessary work and state any test used.

$\lim_{n \to \infty} \frac{2+n^3}{1+2n^3} = \frac{1}{2} \neq 0$

$\boxed{\sum_{n=1}^{\infty} a_n \text{ D by Div Test}}$