Note Title 3_19_08 3/19/2008

$|\alpha\rangle$ ket, labeled by α

represented by a vector

in some coordinates

e.g. $|a\rangle \rightarrow \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ or

or

$\Rightarrow a(x)$

adjoint of $|a\rangle$ is $\langle a| = (|a\rangle)^{+}$

so that $\| |a\rangle \|^2 = \underbrace{\langle a|a\rangle}_{\text{inner product.}}$

outer product $(\vec{x}\vec{x}^T)$ is denoted by $|x\rangle\langle x|$.

recall the geometrical significance of $\vec{x}\vec{x}^T$: this is a projection operator onto $\vec{x}$

$$(\vec{x}\,\vec{x}^T)\vec{y} = \vec{x}\,(\vec{x}^T\vec{y})$$
$$= \vec{x}\,(\vec{x}\cdot\vec{y}) \equiv P_{\vec{x}}\vec{y}$$

usually we use unit vectors for projection so $\hat{x}(\hat{x}\cdot\vec{y})$ is a vector of lenght $\hat{x}\cdot\vec{y}$ pointing along $\hat{x}$.

Notice that $P_x^2 = P_x$ since

$$P_{\vec{x}}\,\vec{y} = \frac{\vec{x}\,\vec{x}^T\vec{y}}{\|x\|^2} = \hat{x}\,\hat{x}^T\vec{y}$$

so $$P_{\vec{x}}^2\, y = P_{\vec{x}}(P_{\vec{x}}\,\vec{y}) = \hat{x}\,\hat{x}^T\left[\hat{x}\,\hat{x}^T y\right]$$

$$= \hat{x}\,\underbrace{(\hat{x}^T\hat{x})}_{\overset{\shortparallel}{1}}\,\hat{x}^T y = \hat{x}\,\hat{x}^T y = P_{\vec{x}}$$

$P^2 = P$ for any projection operator

$$(|x\rangle\underbrace{\langle x|)(|x\rangle}_{\langle x|x\rangle\ =\ 1 \text{ for normalized}}\langle x|)$$

$$= |x\rangle\langle x|$$

if $\vec{x} \in R^N$ and I want to project onto R^N then what is my proj. operator? I_N

The N-dim identity:

$$I_N \vec{x} = \vec{x}$$

Looks trivial, but this tells us that if $|e_i\rangle$ $i=1,N$ is a basis for R^N, then $\{\langle e_i | e_j \rangle = \delta_{ij}\}$

$$\boxed{\sum_{i=1}^{N} |e_i\rangle\langle e_i| = I_N}$$

For continuous orthonormal vectors (i.e., functions)

$$\langle f_x | f_{x'} \rangle = \delta(x - x')$$

Then $$\boxed{\int |f_x\rangle\langle f_x| \, dx = 1}$$

The 2 boxes are called resolutions of the identity

E.g. $|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

labels

could also say $|e_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|e_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

so for any $|s\rangle = a|1\rangle + b|2\rangle$

$$= \begin{pmatrix}a\\b\end{pmatrix}$$

if $|s\rangle$ is normalized then

$$\langle s|s\rangle = 1 = (a^*\langle 1| + b^*\langle 2|)(a|1\rangle + b|2\rangle)$$

$$= |a|^2 + |b|^2 = 1$$

Toy Hamiltonian H:

Let H be a generic Hermitian matrix

$$H = \begin{pmatrix}h & g\\ g & h\end{pmatrix}$$ with h, g real

TDSE in Dirac notation $$i\hbar \frac{d}{dt}|s\rangle = H|s\rangle$$

TISE = $H|s\rangle = E|s\rangle$

E-values of H satisfy char. poly.

$$(h-E)^2 - g^2 = 0 \Rightarrow E_\pm = h \pm g$$

E-vectors are $|s_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$

suppose $|s(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|s_+\rangle + |s_-\rangle\right)$

$$\Rightarrow \quad |s(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i(h+g)t/\hbar}|s_+\rangle + e^{-i(h-g)t/\hbar}|s_-\rangle\right]$$

$$= \frac{1}{\sqrt{2}} e^{-iht/\hbar}\left[e^{-igt/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + e^{igt/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

$$= \frac{1}{2} e^{-iht/\hbar}\begin{bmatrix}2\cos(gt/\hbar)\\ -2i\sin(gt/\hbar)\end{bmatrix}$$

$$= e^{-iht/\hbar}\begin{bmatrix}\cos(gt/\hbar)\\ -i\sin(gt/\hbar)\end{bmatrix}$$

Check that this satisfies

$$i\hbar \frac{d}{dt}|s\rangle = H|s\rangle$$

and $|s(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$