

# Introduction to Fourier optics

Definitions: our signals are the complex fields

t-w space  $f(t)$ ,  $F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \equiv \mathcal{F}\{f(t)\}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega = \mathcal{F}^{-1}\{F(\omega)\}$$

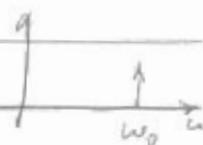
conventions:

1)  $\omega$  not  $\nu$  as frequency variable

2)  $e^{+i\omega t}$  on forward transform

- so that  $\int e^{-i\omega t} dt = \delta(\omega - \omega_0)$

3)  $\frac{1}{2\pi}$  in front of  $\mathcal{F}^{-1}$



X-f<sub>x</sub> space

$$F(f_x, f_y) = \iint_{-\infty}^{\infty} f(x, y) e^{-2\pi i(xf_x + yf_y)} dx dy = \mathcal{F}\{f(x, y)\}$$

$$f(x, y) = \iint_{-\infty}^{\infty} F(f_x, f_y) e^{+2\pi i(xf_x + yf_y)} df_x df_y$$

1)  $f_x$  not  $k_x$  as spatial frequency (Goodman conv.)

2) opposite sign convention from t,  $\omega$

- so that  $\int e^{+ik_x x} dx \rightarrow \delta(f_x - k_x/2\pi)$

3) no  $1/2\pi$  (comes from using  $f_x$  variable)

Why Fourier representation?

physics of linear propagation is usually separable in one domain.

example: dispersive propagation  $E_{out}(w) = E_{in}(w) e^{i\phi(w)}$   
 $\phi(w) = \frac{w}{c} n(w) L$

diffractive propagation:  $E_{out}(x, y, z) = \int \int E_{in}(k_x, k_y) e^{i\phi(k_x, k_y)}$   
 $\phi(k_x, k_y) = -\frac{2\pi}{\lambda} \sqrt{1 - \lambda^2 k_x^2 - \lambda^2 k_y^2} z$

- linear propagation takes place in  $w, k_x, k_y$  space.
- we often measure (or do experiments) in  $t, x, y$  space
  - measure spectrum with spectrometer  $I(w) \propto |E(w)|^2$
  - most measurements are on time-averaged intensity

Research examples:

- pulse compression
- pulse shaping
- spatial filtering
- image processing
- lens design
- light scattering, acousto-optic
- spatio-temporal coupling
- resonators
- guided waves
- Airy waves

Related domains:

- cylindrical coords  $\rightarrow$  Fourier-Bessel, Hankel x form
  - $\rightarrow$  Bessel beams
- Wigner x form: mix  $t-w$  space

## Techniques

- analytic / manual: application of FT pairs, theorems
  - graphical, intuitive understanding of physical systems
- analytic / Mathematica
  - does not easily simplify in terms of xform pairs
  - useful when functions are cumbersome to do by hand.
- numeric / FFT, convolution
  - many real functions cannot be xformed analytically
  - data analysis.
  - modeling.

## Fourier transform pairs

Gaussian

use  $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$  even if  $z$  is complex.

$$f(t) = e^{-t^2/t_0^2}$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{i\omega t} dt$$

complete square in exponent:

$$\frac{t^2}{t_0^2} - i\omega t = \frac{1}{t_0^2} (t^2 - i\omega t t_0^2) = \frac{1}{t_0^2} \left( t - \frac{i\omega t_0^2}{2} \right)^2 + \frac{\omega^2 t_0^2}{4}$$

$$dz = \frac{1}{t_0} dt \rightarrow F(\omega) = e^{-\frac{\omega^2 t_0^2}{4}} \int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= \sqrt{\pi} t_0 \exp\left[-\frac{\omega^2 t_0^2}{4}\right]$$

time-bandwidth product

$$t_0 \cdot \Delta\omega = t_0 \cdot \frac{2}{t_0} = 2$$

$t_0, \Delta\omega = 1/e$  half widths in field.

FWT/1M (more common)

$$I(t) = |F(t)|^2 = e^{-a t^2/t_0^2}$$

$$\text{at } t = T/2 \quad I = \frac{1}{2} = e^{-a/4} \rightarrow \ln 2 = a/4$$

$$a = 4 \ln 2$$

$$f(t) = e^{-2 \ln 2 t^2/t_0^2}$$

converting back to field.

$$\therefore t_0 = T/\sqrt{2 \ln 2} \quad \text{similarly } \Delta\omega = \Delta\omega/\sqrt{2 \ln 2}$$

$$\text{now } T \Delta\omega = 2/\sqrt{2} t_0 \Delta\omega = 4 \ln 2 = \underline{2.77}$$

$$T \Delta f = \frac{4 \ln 2}{2\pi} = \underline{0.44}$$

Parseval's theorem: energy is same in both domains.

$$\int |f(t)|^2 dt = \frac{1}{2\pi} \int |F(\omega)|^2 d\omega \quad \text{note } |Abs[\ ]|^2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int f(t) e^{i\omega t} dt \right] \left[ \int f^*(t') e^{-i\omega t'} dt' \right] d\omega$$

integrate over  $\omega$ :  $\frac{1}{2\pi} \int e^{i\omega(t-t')} d\omega = \frac{1}{2\pi} \int e^{i\omega t} e^{-i\omega t'} d\omega = \delta(t-t')$

$$\int |f(t)|^2 dt = \frac{1}{2\pi} \int dt f(t) \int dt' f^*(t') 2\pi \delta(t-t') \quad \checkmark$$

Example: Gaussian pulses

$$F(\omega) = \sqrt{\pi} t_0 e^{-\omega^2 t_0^2 / 4}$$

$$\frac{1}{2\pi} \int |F(\omega)|^2 d\omega = \frac{1}{2\pi} \pi t_0^2 \int e^{-\omega^2 t_0^2 / 2} d\omega = \frac{1}{2} t_0^2 \frac{\sqrt{2}}{t_0} \sqrt{\pi} = \sqrt{\frac{\pi}{2}} t_0$$

let  $z = \omega t_0 / \sqrt{2}$   $dz = \frac{t_0}{\sqrt{2}} d\omega$   $\rightarrow$

$$\int |f(t)|^2 dt = \int e^{-z^2 / t_0^2} dt = \sqrt{\pi} \frac{t_0}{\sqrt{2}} \quad \checkmark$$

$$z = \sqrt{2} t / t_0, \quad dz = \frac{\sqrt{2}}{t_0} dt$$



Dirac delta function

$$\delta(t) = \infty \text{ at } t=0 \\ = 0 \text{ elsewhere}$$

but  $\int \delta(t) dt = 1$  unit area.

$\delta$  function can be the limit of other functions

$$\lim_{t_0 \rightarrow \infty} \int \text{rect}(t/t_0) = \lim_{t_0 \rightarrow \infty} t_0 \text{sinc}\left(\frac{\omega t_0}{2}\right)$$

$$\text{at } \omega = 0 \rightarrow \infty$$

$$\omega \neq 0 \rightarrow \frac{\sin\left(\frac{\omega t_0}{2}\right)}{\omega/2}$$

here  $\omega/t_0 \rightarrow \infty$  the  $\text{sinc}()$  oscillates rapidly, so that any integral  $\rightarrow 0$

check normalization:

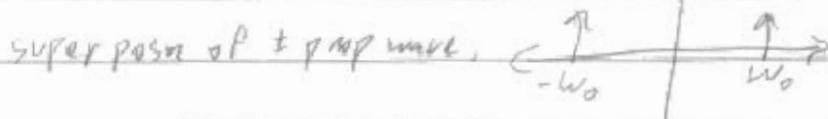
$$\int_{-\infty}^{\infty} t_0 \text{sinc}\left(\frac{\omega t_0}{2}\right) d\omega = 2 \int_{-\infty}^{\infty} \frac{\sin\left(\frac{\omega t_0}{2}\right)}{\omega} d\omega = 2\pi$$

(use contour integration, pole at  $\omega=0$ )

$$\therefore \int \{1\} = 2\pi \delta(\omega) \quad \int \{e^{-i\omega t_0}\} = 2\pi \delta(\omega - \omega_0)$$

$$\int^{-1} \{1\} = \delta(t) \quad \int^{-1} \{e^{i\omega t_0}\} = \delta(t - t_0)$$

by extension,  $\int \{\cos(\omega t)\} = \frac{1}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$



F.T. Theorems:

$$\mathcal{F}\{f(t)\} = F(\omega)$$

scale

$$\mathcal{F}\{f(at)\} = \int f(at) e^{-i\omega t} dt$$

$$\text{let } t' = at \rightarrow \frac{1}{a} \int f(t') e^{-i\frac{\omega}{a} t'} dt'$$

$$= \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

$$\text{if } a < 0, t' = -|a|t \rightarrow \int_{-\infty}^{\infty} f(t') e^{-i\omega\left(\frac{-1}{|a|}t'\right)} dt'$$

$$\mathcal{F}\{f(at)\} = \frac{-1}{|a|} \int_{-\infty}^{\infty} f(t') e^{-i\omega\left(\frac{-1}{|a|}t'\right)} dt'$$

$$= \frac{1}{|a|} F\left(\frac{-\omega}{|a|}\right) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

shift

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0) e^{-i\omega t} dt$$

$$t' = t - t_0 \rightarrow \int_{-\infty}^{\infty} f(t') e^{-i\omega(t'+t_0)} dt'$$

$$= e^{-i\omega t_0} F(\omega)$$

$$\text{linearity } \mathcal{F}\{f_1(t) + f_2(t)\} = F_1(\omega) + F_2(\omega)$$

conjugate

$$\mathcal{F}\{f^*(t)\} = \int f^*(t) e^{-i\omega t} dt$$

$$= \left[ \int f(t) e^{+i\omega t} dt \right]^* = F^*(-\omega)$$

## Symmetry properties of the FT

Any input function will have Re, Im parts

$$f(t) = \text{Re}(f(t)) + i \text{Im}(f(t))$$

each of these can be separated into odd and even parity

$$\text{even } f_e(t) \equiv f_e(-t)$$

$$\text{odd } f_o(t) \equiv -f_o(-t)$$

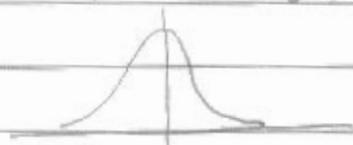
general case:

$$f(t) = f_e(t) + f_o(t)$$

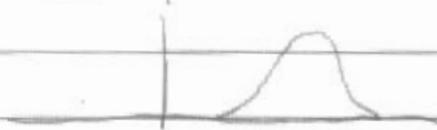
$$f_e(t) = \frac{1}{2} (f(t) + f(-t)) \quad \text{even part}$$

$$f_o(t) = \frac{1}{2} (f(t) - f(-t)) \quad \text{odd part}$$

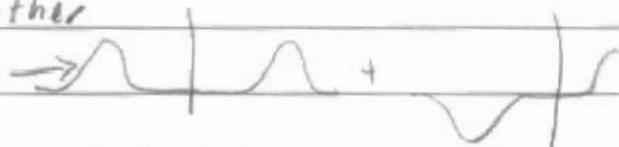
Note that the choice of origin is important:



even



neither



FT

$$F(\omega) = \int [\text{Re}(f(t)) + i \text{Im}(f(t))] [\cos(\omega t) + i \sin(\omega t)] dt$$

we can identify  $\text{Re}(F)$  and  $\text{Im}(F)$

$$\text{Re}(F(\omega)) = \int [\text{Re}(f) \cos \omega t - \text{Im}(f) \sin(\omega t)] dt$$

$$\text{Im}(F(\omega)) = \int [\text{Im}(f) \cos \omega t + \text{Re}(f) \sin \omega t] dt$$

Now we can look at the parity of these components.  
 $\cos(\omega t)$  is even w.r.t both  $t, \omega$   
 $\sin(\omega t)$  is odd.

ex.  $\cos \omega t = \text{Real, even} \rightarrow \text{Real even}$

$\sin \omega t = \text{Real, odd} \rightarrow \text{imag, odd.}$

example: square pulse

$$\left. \begin{array}{l} \text{envelope} \quad \text{rect}(t/t_0) = f_1(t) \\ \text{wave} \quad e^{-i\omega t} = f_2(t) \end{array} \right\} f(t) = f_1(t) f_2(t)$$

$$\begin{aligned} \text{direct } F(\omega) &= \int_{-t_0/2}^{t_0/2} e^{-i\omega t} e^{i\omega_0 t} dt \\ &= \frac{1}{i(\omega - \omega_0)} \left( e^{i(\omega - \omega_0)t} - e^{-i(\omega - \omega_0)t} \right) \end{aligned}$$

shift them:

$$\mathcal{F}\{f(t-t_0)\} = e^{+i\omega t_0} F(\omega)$$

$$\mathcal{F}\{F(\omega - \omega_0)\} = e^{-i\omega_0 t} f(t)$$

here  $\mathcal{F}\{\text{rect}(t/t_0) e^{-i\omega_0 t}\} = t_0 \text{sinc}\left(\frac{(\omega - \omega_0)t_0}{2}\right)$



check

conservation of energy

$$\int |f(t)|^2 dt = \int_{-t_0/2}^{t_0/2} 1 dt = t_0$$

$$\int |F(\omega)|^2 d\omega = \int t_0^2 \text{sinc}^2\left(\frac{(\omega - \omega_0)t_0}{2}\right) d\omega$$

$$= 2\pi t_0$$

in general:

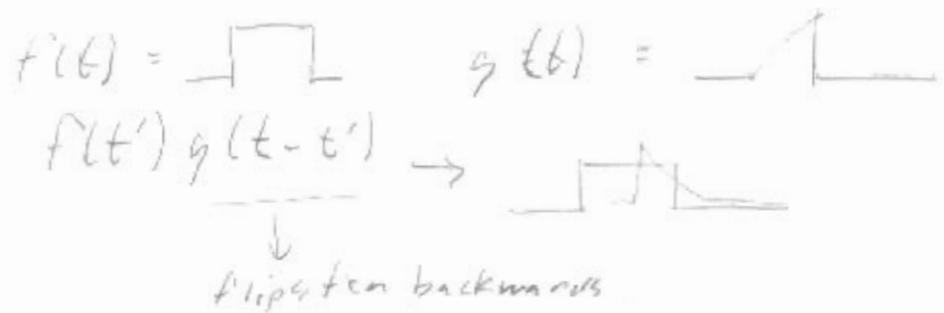
Parseval's theorem:

$$\int |f(t)|^2 dt = \frac{1}{2\pi} \int |F(\omega)|^2 d\omega$$

## Convolution



each pt of  $t$  is integral of product of shifted.



## Correlation

$$h_c(t) \equiv \int f(t') g^*(t+t') dt'$$

no flip

correlation for  $t=0$  measures how similar functions are.

## Autocorrelation

$$h_{ac}(t) = \int f(t') f^*(t+t') dt'$$

transforms:

$$\begin{aligned} \text{first note } h_c &= \int f(-\tau) g^*(t-\tau) d\tau \\ &= -F(-t) \otimes g^*(t) \end{aligned}$$

$$H_c = \int \{ h_c(t) \}$$

## Convolution theorem

define convolution:

$$f(t) \otimes g(t) = \int f(t') g(t-t') dt'$$

$$\text{then:} \quad = \mathcal{F}^{-1} \{ F(\omega) G(\omega) \}$$

proof:

$$\begin{aligned} & \left[ \frac{1}{2\pi} \int F(\omega) e^{-i\omega t} d\omega \right] \left[ \frac{1}{2\pi} \int G(\omega') e^{-i\omega'(t-t')} dt' \right] \\ &= \frac{1}{4\pi^2} \int d\omega F(\omega) \int d\omega' G(\omega') e^{-i\omega t} \underbrace{\int dt' e^{i(\omega'\omega - \omega)t'}}_{2\pi \delta(\omega'-\omega)} \end{aligned}$$

now integrate on  $\omega'$

$$\begin{aligned} &= \frac{1}{2\pi} \int d\omega F(\omega) \int d\omega' \delta(\omega'-\omega) G(\omega') e^{-i\omega t} \\ &= \frac{1}{2\pi} \int d\omega F(\omega) G(\omega) e^{-i\omega t} = \mathcal{F}^{-1} \{ F(\omega) G(\omega) \} \end{aligned}$$

inverse

$$\mathcal{F} \{ f(t) \otimes g(t) \} = \frac{1}{2\pi} F(\omega) \otimes G(\omega)$$

ex. square pulse:

$$\begin{aligned} \mathcal{F} \left\{ \text{rect}(t/t_0) e^{-i\omega_0 t} \right\} &= \frac{1}{2\pi} \left( t_0 \text{sinc}\left(\frac{\omega t_0}{2}\right) \otimes 2\pi \delta(\omega - \omega_0) \right) \\ &= t_0 \text{sinc}\left(\frac{(\omega - \omega_0) t_0}{2}\right) \end{aligned}$$

Convolution Theorem in reverse

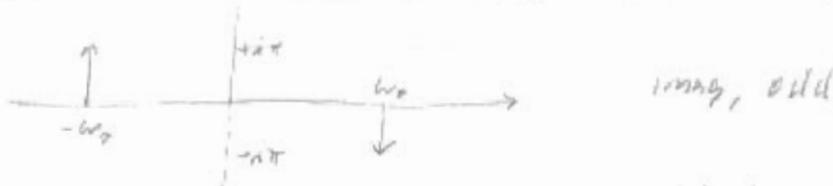
$$\begin{aligned}\mathcal{F}\{f(t)g(t)\} &= \int_{-\infty}^{\infty} e^{-i\omega t} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega') e^{i\omega' t} d\omega' \right] \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega'') e^{i\omega'' t} d\omega'' \right] dt \\ &= \frac{1}{4\pi^2} \iint F(\omega') G(\omega'') \left[ \int e^{i(\omega' + \omega'' - \omega)t} dt \right] d\omega' d\omega'' \\ &= \frac{1}{2\pi} \iint F(\omega') G(\omega'') \delta(\omega' + \omega'' - \omega) d\omega' d\omega'' \\ &= \frac{1}{2\pi} \int F(\omega') G(\omega - \omega') d\omega'\end{aligned}$$

### F.T. examples

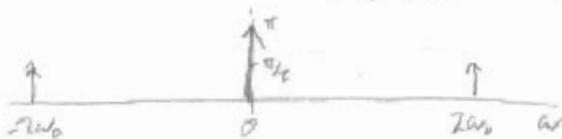
$$\begin{aligned}
 1) \quad \int_{-\infty}^{\infty} \{ \cos \omega_0 t \} &= \frac{1}{2} \int_{-\infty}^{\infty} \{ e^{i\omega_0 t} + e^{-i\omega_0 t} \} \\
 \text{(real, even)} & \\
 &= \frac{1}{2} (2\pi \delta(\omega - \omega_0) + 2\pi \delta(\omega + \omega_0)) \\
 &= \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)
 \end{aligned}$$



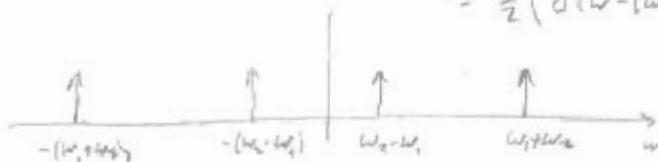
$$\begin{aligned}
 2) \quad \int_{-\infty}^{\infty} \{ \sin \omega_0 t \} &= \frac{1}{2i} \cdot 2\pi (\delta(\omega - \omega_0) - \delta(\omega + \omega_0)) \\
 \text{(real, odd)} & \\
 &= \pi (-\delta(\omega - \omega_0) + \delta(\omega + \omega_0))
 \end{aligned}$$



$$\begin{aligned}
 3) \quad \int_{-\infty}^{\infty} \{ \cos^2 \omega_0 t \} &= \int_{-\infty}^{\infty} \left\{ \frac{1}{4} (e^{2i\omega_0 t} + e^{-2i\omega_0 t} + 1) \right\} \\
 &= \frac{\pi}{2} (\delta(\omega - 2\omega_0) + \delta(\omega + 2\omega_0) + 2\delta(\omega))
 \end{aligned}$$

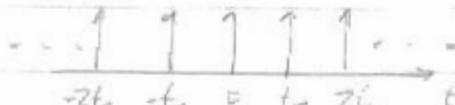


$$\begin{aligned}
 4) \quad \int_{-\infty}^{\infty} \{ \cos \omega_1 t \cos \omega_2 t \} &= \int_{-\infty}^{\infty} \left\{ \frac{1}{4} (e^{i(\omega_1 + \omega_2)t} + e^{-i(\omega_1 + \omega_2)t} + e^{i(\omega_1 - \omega_2)t} + e^{-i(\omega_1 - \omega_2)t}) \right\} \\
 \omega_2 > \omega_1 & \\
 &= \frac{\pi}{2} (\delta(\omega - (\omega_1 + \omega_2)) + \delta(\omega + \omega_1 + \omega_2) + \delta(\omega - (\omega_1 - \omega_2)) + \delta(\omega + (\omega_1 - \omega_2)))
 \end{aligned}$$



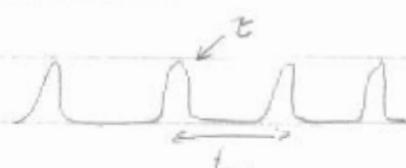
# Array theorem

comb function:

$$\text{comb}(t/t_0) = \sum_{n=-\infty}^{\infty} \delta(t - nt_0)$$


applications:

pulse train:

$$e^{-t^2/w^2} \otimes \text{comb}(t/t_0) \rightarrow$$


Sampling:

$$g(t) \cdot \text{comb}(t/t_0)$$

$t_0 = \text{Sampling period}$



grating

$$[a(x) \otimes \text{comb}(x/x_0)] \cdot \text{rect}(x/D)$$

↑  
groove shape

↑  
periodicity

↑  
 $D = \text{grating length}$

transform:

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nt_0)$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} \mathcal{F}\{\delta(t - nt_0)\} = \sum_{n=-\infty}^{\infty} e^{-i\omega t_0 n}$$

this is actually a comb function:

comb( $\omega$ ) is periodic, so write as Fourier series:

$$f(t) = \sum_n c_n e^{i2\pi n t/t_0}$$

coeff:  $c_n = \frac{1}{t_0} \int_{-t_0/2}^{t_0/2} f(t) e^{-i2\pi n t/t_0} dt \Rightarrow \frac{1}{t_0} \int_{-t_0/2}^{t_0/2} \delta(t) e^{-i2\pi n t/t_0} dt$

let limits  $\rightarrow \pm \infty$

$$c_n = 1/t_0$$

$\therefore$  we can write

$$\text{comb}(t/t_0) = \frac{1}{t_0} \sum_n e^{i2\pi n t/t_0}$$

$$\int \text{comb}(t/t_0) \int e^{i2\pi n t/t_0}$$

$$= \sum_n \frac{2\pi}{t_0} \delta(\omega + \frac{2\pi n}{t_0})$$

$$= \frac{2\pi}{t_0} \text{comb}(\frac{\omega}{2\pi/t_0})$$

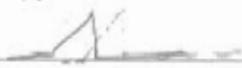
frequency spacing  $\Delta\omega = \frac{2\pi}{t_0} \rightarrow \Delta\nu = \frac{1}{t_0}$

in a laser resonator:  $t_0 = T_{RT} = \frac{2L_0}{c}$  round trip time

## Autocorrelation theorem

$$f_{AC}(\tau) = \int f(t) f^*(t-\tau) dt$$

Physically, this is the time-integrated intensity when two pulses are overlapped with a time delay.



→  $t-\tau$  note: pulse 2 is not flipped

## AC theorem (Wiener-Khinchin)

$$\mathcal{F}\left\{ \int f(t) f^*(t-\tau) dt \right\} = \left| \mathcal{F}\{f(t)\} \right|^2 = |F(\omega)|^2$$

proof:

$$\iint f(t) f^*(t-\tau) dt e^{i\omega\tau} d\tau = \int dt f(t) \left( \int f^*(t-\tau) e^{i\omega\tau} d\tau \right)$$

$$= \int dt f(t) \left[ \int f(t-\tau) e^{-i\omega\tau} d\tau \right]^*$$

could use shift theorem, conjugate theorem, but work manually

let  $t' = t - \tau$   $dt' = -d\tau$  but flip limits  $\int_{+\infty}^{-\infty} dt'$

$$\rightarrow \int dt f(t) \left[ \int f(t') e^{i\omega(t-t')} dt' \right]^* = \int dt f(t) F^*(-\omega) e^{i\omega t}$$

note - sign on  $t'$  (opposite normal xform)  $\therefore \rightarrow$

$$\rightarrow F(\omega) F^*(-\omega)$$

if  $f(t)$  is real then  $F(\omega)$  is even and  $\rightarrow |F(\omega)|^2$