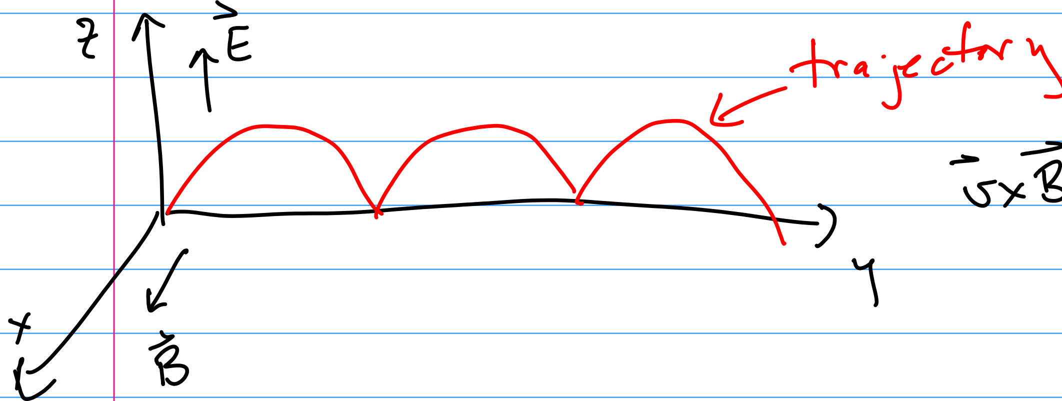
Homework 3 solns

1.) The electric force moves the particle upward. When it gathers speed it experiences a Lorentz force from B which moves it to the right in a circle. As it starts moving downward it slows down which reduces the Lorentz force, finally coming to a stop. The process then begins again.

$$\vec{F} = q\vec{v}\times\vec{B} + q\vec{E} \qquad \vec{v} = v_y\hat{y} + v_z\hat{z}$$



$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & v_y & v_z \\ B & 0 & 0 \end{vmatrix}$$

$$\vec{F} = qE\,\hat{z} + Bv_z\,\hat{y} - Bv_y\,\hat{z} = m\vec{a} = m\left(\frac{dv_y}{dt}\hat{y} + \frac{dv_z}{dt}\hat{z}\right)$$

$$\hat{y}:\ qBv_z = m\frac{dv_y}{dt} \qquad \hat{z}:\ qE - qBv_y = m\frac{dv_z}{dt}$$

Solve coupled ODE's with initial conditions

$$t=0 \quad v_x = v_y = 0 \ \ \&\ \ z = y = 0$$

These conditions specify constants of integration

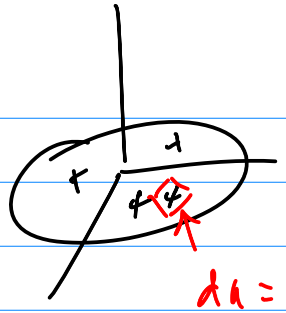
Differential first wrt t and plug into the second to eliminate v_z

Use initial conditions on v_y Then plug into v_z and use initital conditions

Finally, integrate again to find x(t) and y(t)

$$y(t) = \frac{E}{\omega B}\left(\omega t - \sin(\omega t)\right); \quad z(t) = \frac{E}{\omega B}\left(1 - \cos(\omega t)\right); \quad \omega = \frac{qB}{m}$$

2.)



$$\vec{F} = \int \vec{K} \times \vec{B}\, da$$

$$\vec{K} = \sigma \vec{v} = \sigma \omega r \hat{\phi} =$$

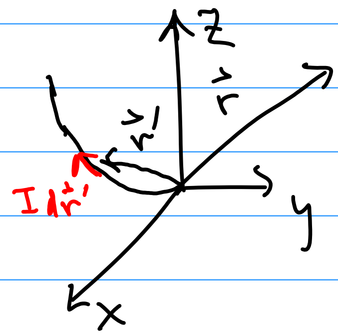
$$= \sigma \omega r \left(-\sin\phi\, \hat{x} + \cos\phi\, \hat{y}\right)$$

$da = r\,dr\,d\phi$

$$\vec{B} = B_x(x,y,z)\,\hat{x} + B_y(x,y,z)\,\hat{y} + B_z(x,y,z)\,\hat{z}$$

limits: ϕ from $0 \to 2\pi$
r from $0 \to R$

3.)



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad \vec{r}' = x'\hat{x} + x'^2\hat{y}$$

$$d\vec{r}' = dx'\,\hat{x} + 2x'\,dx'\,\hat{y}$$

$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0}{4\pi} \frac{I\, d\vec{r}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

limits on x': $0 \to L$

4.) $$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix} = \hat{z}(b-a) \Rightarrow \int \vec{\nabla} \times \vec{v} \cdot d\vec{a} = (b-a)\pi R^2$$

$$\vec{v} \cdot d\vec{r} = (ay\,\hat{x} + bx\,\hat{y}) \cdot (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}) = ay\,dx + bx\,dy$$

$$x^2 + y^2 = R^2 \text{ or } 2x\,dx + 2y\,dy = 0 \Rightarrow dy = -\frac{x}{y}\,dx$$

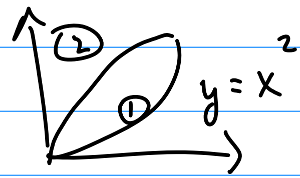
$$\vec{v} \cdot d\vec{r} = ay\,dx + bx\left(-\frac{x}{y}\right)dx = \frac{1}{y}(ay^2 - bx^2)\,dx$$

For upper semicircle $y = \sqrt{R^2 - x^2}$ so $\vec{v} \cdot d\vec{r} = \dfrac{a(R^2 - x^2) - bx^2}{\sqrt{R^2 - x^2}}\,dx$

$$\int \vec{v} \cdot d\vec{r} = \int_{-R}^{R} \frac{aR^2 - (a+b)x^2}{\sqrt{R^2 - x^2}}\,dx = \frac{1}{2}\pi R^2 (b-a)$$

You get the same answer for the lower hemisphere

$$\oint \vec{v}\cdot d\vec{r} = \pi R^2 (b-a) = \int \vec{\nabla}\times\vec{v}\cdot da$$

5.) 

$\vec{r}_{①} = x\hat{x} + x^2\hat{y}$ $\quad d\vec{r}_{①} = dx\,\hat{x} + 2x\,dx\,\hat{y}$

$\vec{r}_{②} = y^2\hat{x} + y\hat{y}$ $\quad d\vec{r}_{②} = 2y\,dy\,\hat{x} + dy\,\hat{y}$

$$\vec{F}\cdot d\vec{r}_{①} = (x+y)\,dx + xy(2x\,dx) = dx\left(x + x^2 + 2x^4\right)$$

$$W_1 = \int \vec{F}\cdot d\vec{r}_{①} = \frac{x^2}{2} + \frac{x^3}{3} + \frac{2}{5}x^5 \Big|_{x=0}^{x=1} = \frac{37}{30}$$

$$\vec{F}\cdot d\vec{r}_{②} = (x+y)\,2y\,dy + x\,dy = dy\left(2xy + 2y^2 + xy\right) \qquad = y^2$$

$$W_2 = \int \vec{F}\cdot d\vec{r}_{②} = \int_1^0 dy\left(3y^3 + 2y^2\right) = -\frac{17}{12}$$

$$W_{tot} = -11/60$$

6.)

$$\frac{\hat{r}}{r^2} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}$$

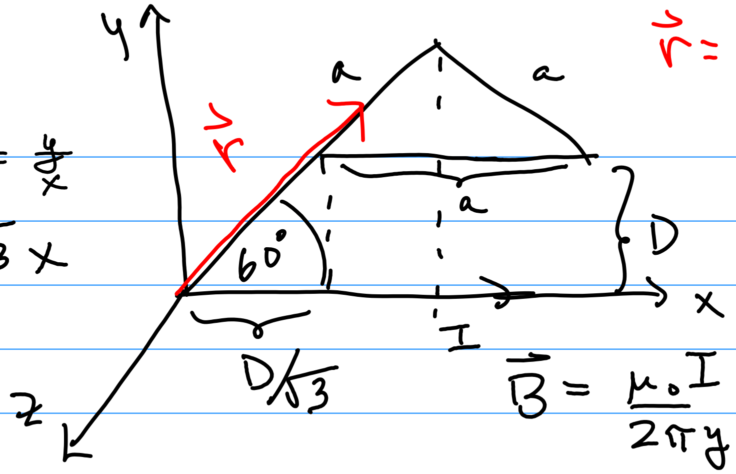
$$\vec{\nabla}\times\frac{\hat{r}}{r^2} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{r_x}{r^3} & \frac{r_y}{r^3} & \frac{r_z}{r^3} \end{vmatrix}$$

Plug into Mathematica

7.)

$$\tan 60 = \frac{y}{x}$$

$$y = \sqrt{3}\,x$$



$$\vec{r} = x\hat{x} + \underbrace{y}_{\sqrt{3}\,x}\hat{y} + \underbrace{z}_{0}\hat{z}$$

$$d\vec{r} = dx\,\hat{x} + \sqrt{3}\,dx\,\hat{y}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi y}\hat{k} \qquad d\vec{F} = I\,d\vec{r} \times \vec{B}$$

$$d\vec{F} = I\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx & \sqrt{3}\,dx & 0 \\ 0 & 0 & \frac{\mu_0 I}{2\pi y} \end{vmatrix} = \frac{\mu_0 I^2}{2\pi y}\left(-dx\,\hat{y} + \sqrt{3}\,dx\,\hat{x}\right)$$

The x component of this force will cancel that from the other diagonal segment while the y component will add.

$$F_y = -\frac{2\mu_0 I^2}{2\pi}\int_{D/\sqrt{3}}^{\frac{D}{\sqrt{3}}+\frac{a}{2}} \frac{1}{\underbrace{y}_{\sqrt{3}\,x}}\,dx$$

Force from bottom segment is

$$\int I\,d\vec{r} \times B = \int_0^a I\,dx\,\hat{x} \times \frac{\mu_0 I}{2\pi D}\hat{z} = \frac{\mu_0 I^2}{2\pi}\,\frac{a}{D}\left(-\hat{y}\right)$$

8.) see lecture notes