The following document summarizes important results associated with the second-order linear non-homogeneous ordinary differential equation with variable coefficients,

$$a(t)y'' + b(t)y' + c(t)y = f(t), \quad a, b, c, f \in C^n(\mathbb{R}),\ n \in \mathbb{Z}, \tag{1}$$

with the assumption that the reader has had a typical calculus sequence leading to an introductory course in ordinary differential equations (ODE). The goal of this document in to record important theory and formula applicable to equation (1) as well as a few 'standard' solution techniques. This document will have the following organization:

1. (1) The General Linear Theory
2. (2) Overview of Second-Order Linear Theory
3. (3) Results for Constant Coefficient Problems
   - (a) Homogeneous Problems
   - (b) Nonhomogeneous problems
4. (4) Results for Variable Coefficient Problems
   - (a) Power-Series
   - (b) Frobenius method
5. (5) Integral Transform Methods
   - (a) Laplace Transform of IVP's
   - (b) Convolutions and Green's Functions

1. The General Linear Theory

This section summarizes the important facts of linear equations of arbitrary order. To do this we follow two equivalent perspectives with the understanding that the reader has had some amount of exposure to linear algebra. The two perspectives are:

(1) Scalar n^{th} order linear ODEs.
(2) Systems of n-many first order ODEs.

1.1. Scalar n^{th} order linear ODEs.

In this section we consider the general theory associated with the n^{th}-order linear nonhomogeneous ODE with variable coefficients.

$$L\left[y\right] = \sum_{i=0}^{n} a_i(t)\frac{d^i y}{dt^i} = a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + a_{n-2}(t)\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_0(t)y = f(t), \tag{2}$$

where $a_0, \ldots, a_n, f \in C^m\left(\alpha, \beta\right),\ m \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$.

Notes:

- Recall that y is the dependent variable and t is the independent variable.
- The equation is linear since it is a linear combination of the dependent variable and its derivatives. The form that t takes in the equation is inconsequential.
- We call L a *linear differential* operator whose application on a function y results in the differential equation. In this way (2) takes the form $L[y] = f$, which reminds us of $\mathbf{Ax} = \mathbf{b}$.
- If $f(t) = 0$ for all $t \in (\alpha, \beta)$ then we say that the equation is homogeneous.
- In general, the homogeneous linear ODE defines an infinite family of solutions. This can be seen since the general solution of (2) takes the form of an arbitrary linear combination given by,

$$y_h(t) = \sum_{i=1}^{n} c_i y_i(t) \text{ where } c_i \in \mathbb{R} \text{ for } i = 1, 2, 3 \ldots \tag{3}$$

  That is, (3) is the general solution to $L[y] = 0$. If $c_i = 0$ for $i = 1, 2, 3, \ldots$ then we say that $y_h(t)$ is the trivial solution. Otherwise, we say that $y_h(t)$ is a non-trivial solution. This again reminds us of the fundamental solution sets to $\mathbf{Ax} = \mathbf{0}$.

- If $f(t)$ is not the zero-function then we say that the equation is non-homogeneous and it can be shown that the general solution to (2) is of the form,

(4) $$y(t) = y_h(t) + y_p(t),$$

where $y_h(t)$ is nothing more than (3) and $y_p(t)$ is a particular solution to the non-homogeneous problem (2). That is, the solution to $L[y] = f$ are shifted solutions to $L[y] = 0$ where the 'shift' is given by $y_p(t)$. This should, again, remind us of how solutions to $\mathbf{Ax} = \mathbf{b}$ are given by the form $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, where $\mathbf{x}_h$ are solutions to $\mathbf{Ax} = \mathbf{0}$, which have been shifted off the origin by an amount $\mathbf{x}_p$.

- To find a unique solution to (2) initial conditions must be specified. That is, (2) and,

(5) $$y(0) = y_0,\ y'(0) = y_0',\ y''(0) = y_0'', \cdots, y^{(n-1)}(0) = y_0^{(n-1)},$$

define an initial value problem (IVP). Application of (5) to (4) produces exactly one solution out of the infinite family of solutions.

The key-point to take from this recall is that the solution to (2) must take on the form of (4), which is, in fact, an infinite family of solutions. To specify a single solution from this infinite family requires an initial condition and (2) together with (5) constitutes an initial-value problem. Now the problem is,

- Given (2) and (5) how does one find the unique (4)?

Generally, this can be very difficult and though much is known we must localize our efforts to important sub-classes pertinent to physical problems. Before we address this we consider how this is related to linear algebra.

1.2. **Systems of n-many first order ODEs.** It can be shown that the general n^{th}-order ODE, (2), can be rewritten as an $n \times n$ linear system of first-order ODE's. Once this is done, results of linear algebra can be applied that give a general picture of the solution for when $a_i(t) = a_i \in \mathbb{C}$ for $i = 0, 1, 2, \ldots$. In general, less can be said for the case where the coefficients are non-constant and is still an active area of mathematical research.

Theorem 1. *Let $L[y] = f$ be the general n^{th}-order linear ODE given by (2) with initial-condition (5). There exists a variable transformation such that (2) and (5) can be written as,*

(6) $$\frac{d\boldsymbol{Y}(t)}{dt} = \boldsymbol{A}(t)\,\boldsymbol{Y}(t) + \boldsymbol{F}(t), \quad \boldsymbol{Y}(0) = \boldsymbol{Y}_0, \text{ where } \boldsymbol{Y}, \boldsymbol{Y}_0, \boldsymbol{F} \in \mathbb{C}^n, \quad \boldsymbol{A} \in \mathbb{C}^{n\times n},$$

which is a n-dimensional system of linear first-order IVP's.

Explicitly, the previous vectors are of the form,

(7)
$$\mathbf{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \mathbf{Y}_0 = \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ \vdots \\ y_n(0) \end{bmatrix}, \mathbf{F}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{bmatrix}, \quad y_i(t), f_i(t) \in C^m(\alpha,\beta), i = 1, 2, 3, \ldots, n$$

while the matrix is of the form,

(8) $$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \cdots & a_{2n}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) & \cdots & a_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & \cdots & a_{nn}(t) \end{bmatrix}, a_{ij}(t) \in C^m(\alpha,\beta), \quad \begin{matrix} i = 1, 2, 3, \ldots, n, \\ j = 1, 2, 3, \ldots, n, \end{matrix}$$

where $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$.

Proof. Consider, TBD □

Notes:

- For (6) we adopt similar language. If $\mathbf{F}(t) = 0$ for all t then we say the system is homogeneous. Otherwise we say the system is non-homogeneous.
- We also have similar results, which state the the solution to (6) must take the form,

$$\mathbf{Y}(t) = \mathbf{Y}_h(t) + \mathbf{Y}_p(t), \tag{9}$$

  where $\mathbf{Y}_h(t)$ is the solution to the homogeneous problem and $\mathbf{Y}_p(t)$ is a particular solution to the non-homogeneous problem.
- If the coefficient matrix $\mathbf{A}(t)$ is constant in time, $\mathbf{A}(t) = \mathbf{A} \in \mathbb{C}^{n\times n}$ then we say the problem is a constant linear problem or constant coefficient problem and in this case we have very general results.

Theorem 2. Constant Linear Systems
If (6) is such that $\boldsymbol{A}(t)$ is constant in time then the homogeneous solution to (6) is of the form,

$$\boldsymbol{Y}(t) = e^{\boldsymbol{A}t}\,\boldsymbol{Y}_0. \tag{10}$$

Moreover, if the eigenvector problem for $\boldsymbol{A}$ leads to n-linearly independent eigenvectors then (10) takes the form,

$$\boldsymbol{Y}(t) = \sum_{i=1}^{n} c_i\,\boldsymbol{Y}_i(t), \tag{11}$$

where $c_i \in \mathbb{C}$ and $\boldsymbol{Y}_i(t) = \boldsymbol{Y}_i e^{\lambda_i t}$ is an eigenfunction defined in terms of the eigenvector $\boldsymbol{Y}_i$ and eigenvalue λ_i.

Proof. Assume that $\mathbf{A}$ is similar to a diagonal matrix. [1] □

The previous result tells us that the solution to a linear, constant coefficient, homogeneous ODE can be easily written in the eigenfunction basis of the coefficient matrix. This is important to note when making the connection to more general linear operators.[2] Though, this result is general to arbitrary finite order the common framework takes the form of (2) and it focuses on analytic results for the more general non-homogeneous variable coefficient problem.

2. Overview of Second-Order Linear Theory

Since problems of the following type typically derive from force relations, the general problem of (2) commonly take the form,

$$a(t)y'' + b(t)y' + c(t)y = f(t), \tag{12}$$

where Lagrange's 'prime' notation for derivatives has been used for convenience. The coefficients and inhomogeneity are as before but the index notation has been suppressed. Some general facts associated with (12) are,

- The general solution to (12) must takes the form,

$$y(t) = y_h(t) + y_p(t), \tag{13}$$

  where y_h is the solution to (12) when $f(t) = 0$ for all t and y_p is any solution to the (12).

[1]This is not particularly restrictive assumption since it can be shown that the set of all matrices that do not have a diagonalization forms a set of measure zero. This is to say that the set of all matrices that can be diagonalized is dense in the space of square matrices and thus are all that is needed from the perspective of calculus.

[2]The linear operator $\partial_t + \triangle$ defines a infinite dimensional eigenfunction basis, which means that linear combinations will have the added calculus of infinite-series.

- The homogeneous solution to (12) has the form,

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t), \tag{14}$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary constants and $y_1(t)$ and $y_2(t)$ are two linearly independent solutions to the homogeneous problem.

- Two functions are linearly independent if and only if they have a non-zero Wronskian determinant. That is, y_1 and y_2 are linearly independent functions if and only if

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \neq 0. \tag{15}$$

This is generally done by inspection since, from this, it can be shown that two functions are linearly independent if they are not constant multiples of each other.

Now that we know the general form solutions to (12) must take we ask the following question.

- Given (12), how do we find the general solution?

This is not always an easy task but the following theorems will reduce the problem to finding a single solution to the homogeneous problem. Though this is a nice result the remaining parts of the general solution are written in terms of integrations, which are not always possible to do in closed form. Details aside, we have:

Theorem 3. *If $y_1(t)$ is a solution to the homogeneous form of (12) then $y_2(t) = c(t)y_1(t)$ is a second linearly independent solution to (12), where,*

$$c(t) = \int \frac{u(t)}{[y_1(t)]^2} dt, \;\; u(t) = e^{-\int \frac{b(t)}{a(t)} dt}. \tag{16}$$

Proof. Assume that $y_2(t) = c(t)y_1(t)$. Then using this in the homogeneous form of (12) gives, TBD □

Theorem 4. *If $y_h(t)$ is known for (12) then the particular solution must take the form,*

$$y_p(t) = y_2 \int \frac{y_1(t)f(t)}{W(t)} dt - y_1 \int \frac{y_2(t)f(t)}{W(t)} dt, \tag{17}$$

where $W(t)$ is the Wronskian determinant found from the homogeneous solution.

Proof. Assume that $y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$. Then using this in (12) gives, TBD □

These three theorems show that if one can find a single homogeneous solution to (12) then it is possible to find the general solution to the non-homogeneous equation. In general, finding the first homogeneous solution is non-trivial and can require a good deal of analytic work. Performing the final integrations leading to the general solution is added work and integrals may not have known closed form solutions. For this reason we move on to the study of constant coefficient problems where a complete theory is known.

3. Results for Constant Coefficient Problems

Just as in the constant linear system (6) methods for solving,

$$ay'' + by' + cy = f(t), \;\; a, b, c \in \mathbb{C}, \tag{18}$$

are known in detail. The reason for this is that the solution to the corresponding homogenous problem is totally understood and tractable by hand. The methods outlined here generalize to higher-order problems but lead to algebraic problems that require computational tools.[3] We begin

[3] If we consider this from the perspective of (11) then it is clear from linear algebra that an n-dimensional system leads to a n^{th}-degree characteristic polynomial and though we know that this polynomial must have n-many roots, counting multiplicity of course, past degree four it is known that they cannot, generally, be found by current analytic techniques. This result is known as Abel-Ruffini theorem and gives a point where numerical approximation must take over.

with the homogeneous problem and move onto methods used to solve the non-homogeneous problem, keeping in mind that (17) provides a general method if the quicker/elementary techniques fail.

3.1. **Homogeneous Problems.** Solving ODE's has relied heavily on the use of guessing. If you guess that $y_g(t)$ is a solution to an ODE then this assumption can be verified by direct substitution of the guess into the ODE. If equality is maintained then the guess is correct and if it isn't then the guess is incorrect. It turns out that the ODE,

$$ay'' + by' + cy = 0, \tag{19}$$

always leads to the same guess. The idea is that (19) is asking if there is exists a function such that differentiation of that function returns a constant multiple of the function itself. [4] The function with this property is the exponential function. [5] Substituting our guess, $y(t) = e^{\lambda t}$, into (19) gives,

$$\begin{aligned} ay'' + by' + cy &= a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} && (20)\\ &= (a\lambda^2 + b\lambda + c)e^{\lambda t} && (21)\\ &= 0. && (22) \end{aligned}$$

Since the exponential function is never zero we can divide it out of the equation and obtain,

$$a\lambda^2 + b\lambda + c = 0, \tag{23}$$

which is called the characteristic polynomial for the ODE.[6] Since the polynomial is quadratic it can be solved in using the quadratic equation to get,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{24}$$

When $b^2 - 4ac \neq 0$ the characteristic polynomial defines two linearly independent solutions to (19) and thus the complete homogeneous solution to the problem. When $b^2 - 4ac = 0$ then λ is a repeated-root and the only immediate solution is $y_1(t) = e^{-b/2a}$. However, a second solution can be found by the use of Theorem 3.[7] These results are summarized in the following table.

[4] Just think of the case where $a = 0, b = 1, c = -1$, which gives $y' = y$ whose solution is clearly $y(t) = e^t$. If we continue this line of thinking to $a = 1, b = -1, c = -2$ then the equation is $y'' = y' + 2y = \alpha y + 2y = (\alpha + 2)y$, which is again in the same form.

[5] Sine and cosine functions also share this property after more derivatives are taken. However, we won't worry about this since Euler's formula will find these functions for us.

[6] This is the same polynomial one would find if you studied the eigenvalue problem of the corresponding 2×2 system of first-order ODEs.

[7] Direct substitution of $y_1(t) = e^{-b/2a}$ into Theorem 3 proves the common result that $y_2(t) = ty_1(t)$.

Discriminant	Solutions	Homogeneous Solution	Definitions
$b^2 - 4ac > 0$	$y_1(t) = e^{\lambda_1 t}$ $y_2(t) = e^{\lambda_1 t}$	$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$	$c_1, c_2 \in \mathbb{C}$ $\lambda_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ $\lambda_2 \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$
$b^2 - 4ac < 0$	$y_1(t) = e^{\lambda_1 t}$ $y_2(t) = e^{\lambda_1 t}$	$\begin{aligned} y_h(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ &= b_1 e^{\alpha t} \cos(\beta t) + b_2 e^{\alpha t} \sin(\beta t) \end{aligned}$	$c_1, c_2, b_1, b_2 \in \mathbb{C}$ $b_1 = c_1 + ic_2,\ b_2 = c_1 - ic_2$ $\lambda = \alpha \pm \beta i$ $\alpha = \dfrac{-b}{2},\ \beta = \dfrac{\sqrt{4ac - b^2}}{2a}$
$b^2 - 4ac = 0$	$y_1(t) = e^{\lambda t}$ $y_2(t) = te^{\lambda t}$	$y_h(t) = c_1 e^{\lambda t} + c_2 te^{\lambda t}$	$c_1, c_2 \in \mathbb{C}$ $\lambda = \dfrac{-b}{2a}$

3.2. **Non-homogeneous problems.** The previous section completely solves the homogeneous form of

$$ay'' + by' + cy = f(t), \tag{25}$$

and (17) gives a general method to find the particular solution for some $f(t)$. However, these integrals can be costly to evaluate and for simple $f(t)$ guessing can again provide an easy route to the general solution. In fact, there exists a systematic guessing procedure if $f(t)$ contains sums or products of sine/cosine/exponential/polynomial functions. This procedure is called often called 'the method of undetermined coefficients' and relies on the idea that if $f(t)$ contains one of these types of functions then $y_p(t)$ must contain some amount of the same function, which drives the form of the 'guess.' This guess can then be checked by direct substitution and from this the unknown coefficients can be found. The only time this 'guess' can be incorrect is when terms in $y_p(t)$ are proportional to either $y_1(t)$ or $y_2(t)$.[8] If this is the case then the offensive guess must be multiplied by t until the issue is resolved.[9] The following table, where all capital letters represent unknown constant that must be found by direct substitution, outlines this guessing method.

[8] It is nice to know that even if you have guessed wrong the method will fail and not give rise to false positives.

[9] This is similar to the repeated-root case for the homogeneous problem and can all be formally justified by the use of (17).

$f(t)$: Inhomogeneous Term	$y_p(t)$: Guess for the Particular Solution
$P_n(t) = a_n t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_0$	$A_n t^n + A_{n-1}t^{n-1} + A_{n-2}t^{n-2} + \cdots + A_0$
$e^{\alpha t}$	$Ae^{\alpha t}$
$\sin(\beta t)$ or $\cos(\beta t)$	$A\sin(\beta t) + B\cos(\beta t)$
$P_n(t)e^{\alpha t}\sin(\beta t)$	$\left(A_n t^n + A_{n-1}t^{n-1} + A_{n-2}t^{n-2} + \cdots + A_0\right) e^{\alpha t}\sin(\beta t) +$ $+ \left(B_n t^n + B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_0\right) e^{\alpha t}\cos(\beta t)$
$P_n(t)e^{\alpha t}\cos(\beta t)$	$\left(A_n t^n + A_{n-1}t^{n-1} + A_{n-2}t^{n-2} + \cdots + A_0\right) e^{\alpha t}\sin(\beta t) +$ $+ \left(B_n t^n + B_{n-1}t^{n-1} + B_{n-2}t^{n-2} + \cdots + B_0\right) e^{\alpha t}\cos(\beta t)$
$P_n(t) + e^{\alpha t} + \sin(\beta t)$	$A_n t^n + A_{n-1}t^{n-1} + A_{n-2}t^{n-2} + \cdots + A_0 +$ $+ Be^{\alpha t} + C\sin(\beta t) + D\cos(\beta t)$
$P_n(t) + e^{\alpha t} + \cos(\beta t)$	$A_n t^n + A_{n-1}t^{n-1} + A_{n-2}t^{n-2} + \cdots + A_0 +$ $+ Be^{\alpha t} + C\sin(\beta t) + D\cos(\beta t)$

This procedure is complete and now implies a general solution can always be found for,

$$ay'' + by' + cy = f(t), \tag{26}$$

where $f(t)$ is a form found in the table above. Often the ODE is in such a form, but sometimes it is not. If $f(t)$ is a more complicated function the one can appeal to (17). However, the complication tends to arise in the coefficients. The next section will outline a method that can be applied to, (12), the more general second-order ODE.

4. Results for Variable Coefficient Problems

If the coefficients to the ODE are not constant then the previous guess of an exponential form is no longer valid. It is possible to generalize the guess to a general power-series and attempt to find any unknown constant by direct substitution. This method is elaborate but opens the door to a wider class of functions defined by ODE's. It is best to begin with,

$$a(t)y'' + b(t)y' + c(t)y = 0, \tag{27}$$

which is the homogeneous form of (12). The focus is on finding one solution to the ODE since it is known that from this all other terms in the general solution can be found. The primary difference between this equation and a constant coefficient equation is that $a(t)$ may sometimes be zero for particular values of t. These points are called *singular points* and marks a fundamental change to the order of the equation, which defines the form of the general solution. For this reason it is typical to divide the study of (27) into two sections, which studies solutions first about non-singular points and then about singular points.

4.1. **Power-Series.** To simplify matters first assume that for, (27), $a(t) \neq 0$ for all t. Now make the very general guess that the solution to this ODE takes the form,

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \tag{28}$$

where the coefficients a_i for $i = 0, 1, 2, \dots$ are unknown. This assumption can be checked via direct substitution and requires derivatives of (28) to be taken. Mathematically, one should consider the convergence properties of the infinite-series so that term-by-term differentiation can be done. However, this is not a grave concern because if it were not true then the method would fail. It is best to let the failure occur in action. These derivatives then are,

$$y'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n, \tag{29}$$

$$y''(t) = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n, \tag{30}$$

where some series can be more useful than others. It is not straight-forward to proceed without knowledge of $a(t), b(t), c(t)$. The following examples are used to illustrate the completion of this method.

Example 1: Constant Coefficient Linear Problem
Consider the differential equation,

$$y'' + y = 0, \tag{31}$$

where the solution is known to be $y(t) = c_1 \cos(t) + c_2 \sin(t)$. This can also be found with the more elaborate guess of power-series. Replacing the derivatives of y with their power-series gives,

$$y'' + y = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n \tag{32}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=0}^{\infty} a_n t^n \tag{33}$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] t^n \tag{34}$$

$$= 0. \tag{35}$$

Since the previous series is zero for all t the coefficients must be zero.[10] This leads to a recurrence relation, $(n+2)(n+1)a_{n+2} + a_n = 0$, on the coefficients, which is typically written in the form,

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \dots. \tag{36}$$

The previous relation implies that any coefficient is known in terms of a coefficient that proceeded it. In fact, in this case, it connects the even coefficients to even coefficients and odd to odd. To see this consider the recurrence relation for specific n-values.

[10]Think about it like this. Let $p(t) = at^2 + bt + c$ such that $p(t) = 0$ for all t. The quadratic equation tells us that $p(t)$ may have, at most, two roots or values for t where $p(t)$ is zero. Requiring that this polynomial is zero for all values of t implies that our parabola isn't a parabola but instead the zero-function, which means that $a = b = c = 0$. The power-series argument is the same but for an arbitrarily long polynomial.

Even Coefficients	Odd Coefficients
$a_2 = \frac{-a_0}{2 \cdot 1}$	$a_3 = \frac{-a_1}{3 \cdot 2 \cdot 1}$
$a_4 = \frac{-a_4}{4 \cdot 3} = -\frac{-a_0}{4 \cdot 3 \cdot 2 \cdot 1}$	$a_5 = \frac{-a_3}{5 \cdot 4} = -\frac{-a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$
$\vdots$	$\vdots$
$a_{2k} = \frac{(-1)^k a_0}{(2k)!}$	$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}$

The last entries are the generalization of the pattern where the following are noticed:[11]

(1) The even coefficients all recurse back to a_0 and the odd coefficients recurse back to a_1. Nothing more can be said about these two coefficients.

(2) As the recursion is applied to get back to a_0 or a_1 a sign alternation occurs. This is characterized by $(-1)^k$.

(3) As the recursion is applied a product forms in the denominator. This product is precisely the factorial of the coefficient's subscript.

Using this information we now have the following,

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n && (37)\\
&= \sum_{n=0}^{\infty} a_{2n} t^{2n} + \sum_{n=0}^{\infty} a_{2n+1} t^{2n+1} && (38)\\
&= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} && (39)\\
&= a_0 \cos(t) + a_1 \sin(t), && (40)
\end{aligned}
$$

and note that the recurrence relation has lead to the Taylor coefficients for cosine and sine and the unknown constants a_0 and a_1 now play the role of c_1 and c_2 from the homogeneous solutions from above.

So, we have now have a method that agrees with previous results and can be adapted to variable coefficients, which can be seen in the next example. It should be noted that, though this method is not particularly easy, once the recurrence relation is found the series can be evaluated to arbitrary order giving the user as much decimal precision as needed.

Example 2: Variable Coefficient Linear Problem

[11] A generalized pattern is not always possible to find. However, it should be noted that once the recursion relation is found the job is done. Every coefficient can be known in terms of ones that precede connecting coefficients to a_0 and/or a_1. These originators are actually the same as the unknown constants c_1 and c_2 found in the homogeneous solution and can be found via initial conditions. Thus, initial conditions will allow the evaluation of the infinite series to arbitrary decimal precision, which is good enough for tunnel work, as they say.

Consider the differential equation,

$$y'' + ty = 0, \tag{41}$$

which is called Airy's equation and is used in models pertaining to ... TBD

4.2. Frobenius method.

5. Integral Transform Methods

5.1. Laplace Transform of IVP's.

5.2. Convolution and Green's Functions.