1. Find the largest natural number m such that $n^3 - n$ is divisible by m for all $n \in \mathbb{N}$. Prove your assertion.

2. Let the numbers x_n be defined as follows:

$$x_1 := 1, x_2 := 2, \text{ and } x_{n+2} := \frac{1}{2}(x_{n+1} + x_n) \text{ for all } n \in \mathbb{N}.$$

Using induction, show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

6.3.21 Let C be the set of all real functions that are continuous on the closed interval $[0, 1]$. Define the function $A : C \to \mathbb{R}$ as follows:

$$\text{For each } f \in C, A(f) = \int_0^1 f(x)\,dx.$$

Is the function A an injection? Is it a surjection? Justify your conclusions.

6.3.22 Let $A = \{(m, n) \mid m \in \mathbb{Z}, n \in \mathbb{Z}, \text{ and } n \neq 0\}$. Define the function $f : A \to \mathbb{Q}$ as follows:

$$\text{For each } (m, n) \in A, f(m, n) = \frac{m+n}{m}.$$

Is the function f an injection? Is it a surjection? Justify your conclusions.

6.6.5 In addition to considering how a function acts on elements of a given set, we can also discuss how the function acts on subsets of the domain or codomain. For example, let $f : S \to T$ and let $A \subseteq S$. Then we can define

$$f(A) = \{f(x) \mid x \in A\}$$

In this case, $f(A)$ is referred to as the image of A under f.
Prove: $f(A \cup B) = f(A) \cup f(B)$.