Solutions

3.1, 12. Determine if each of the following statements is true or false. Provide a counterexample for statements that are false and provide a complete proof for those that are true.

a. For all real numbers x and y, $\sqrt{xy} \leq \dfrac{x+y}{2}$.

The proposition is false. As a counterexample consider $x = -2$ and $y = -2$. Then we have

$$\sqrt{(-2)(-2)} = \sqrt{4} = 2 > \frac{(-2)+(-2)}{2} = -2$$

b. For all real numbers x and y, $xy \leq \left(\dfrac{x+y}{2}\right)^2$.

Proof. Let $x, y \in \mathbb{R}$ and consider

$$\begin{aligned} 0 &\leq (x-y)^2 = (x^2 - 2xy + y^2) \\ \Rightarrow 4xy &\leq (x^2 + 2xy + y^2) = (x+y)^2 \\ \Rightarrow xy &\leq \left(\frac{x+y}{2}\right)^2 \end{aligned}$$

□

c. For all nonnegative real numbers x and y, $\sqrt{xy} \leq \dfrac{x+y}{2}$.

Proof. Let $x, y \in \mathbb{R}^+$ and note that

$$x \leq y \Rightarrow x - y \leq 0 \Rightarrow (x+y)(x-y) \leq 0 \Rightarrow x^2 - y^2 \leq 0 \Rightarrow x^2 \leq y^2$$

Then

$$\sqrt{xy} \leq \frac{x+y}{2} \Rightarrow xy \leq \left(\frac{x+y}{2}\right)^2$$

The remainder of the proof follows from part (b) above □

3.1, 13. Use one of the true inequalities in Exercise (12) to prove the following proposition.

Proposition 1. *For each real number a, the value of x that gives the maximum value of $x(a-x)$ is $x = \frac{a}{2}$.*

Proof. Using the proposition given in (b) above,

$$x(a-x) \le \frac{x+(a-x)}{2} = \left(\frac{a}{2}\right)^2$$

Therefore, $(\frac{a}{2})^2$ is the maximum value of $x(a-x)$. Solving the equation, $x(a-x) = (\frac{a}{2})^2 \Rightarrow 4x^2-4ax+a^2 = (2x-a)^2 = 0$. Thus, the maximum value of $x(a-x)$ occurs when $x = \frac{a}{2}$. □

3.1, 15. Evaluation of proofs

a. The proposition is true but this proof needs clarification.

Proof. Let m be an even integer. Then there exists a $k \in \mathbb{Z}$ such that $m = 2k$. Now consider $5m+4 = 5(2k)+4 = 10k+4 = 2(5k+2)$. Therefore, since $5k+2 \in \mathbb{Z}$, $5m+4$ is an even integer. □

b. The proposition is true and the proof is correct.

c. The proposition is false. For a counterexample, consider $a = 6$, $b = 2$ and $c = 3$.

3.2, 4. Given the statement,

For all positive real numbers a and b, if $\sqrt{ab} \neq \frac{a+b}{2}$ then $a \neq b$.

a. The contrapositive is given by

For all positive real numbers a and b, if $a = b$ then $\sqrt{ab} = \frac{a+b}{2}$.

b. This statement is true and given by

Proof. Let a, b be positive real numbers such that $a = b$. Then $\sqrt{ab} = \sqrt{a^2} = |a| = a$. Also, $\frac{a+b}{2} = \frac{a+a}{2} = a$.
Thus, $\sqrt{ab} = \frac{a+b}{2}$. □

3.2, 18. Evaluation of proofs

a. The proposition is true. However, the proof needs clarification.

Proof. Let $m \in \mathbb{Z}$ such that m is odd. By definition, there exists an integer k such that $m = 2k+1$. Now consider $m+6 = (2k+1)+6 = 2k+7 = 2(k+3)+1$. By the closure property of integers, $k+3 \in \mathbb{Z}$.
Thus, by definition, $m+6$ is odd. □

b. The proposition is true, but the proof needs clarification.

Proof. We will prove using the contrapositive which states
For integers m and n, if m is odd and n is odd, then mn is odd.
Let $m, n \in \mathbb{Z}$ such that m and n are both odd. Then there exist $j, k \in \mathbb{Z}$ such that $m = 2j+1$ and $n = 2k+1$. Now consider

$$mn = (2j+1)(2k+1) = 4jk + 2j + 2k + 1 = 2(2jk + j + k) + 1$$

Since $2jk + j + k \in \mathbb{Z}$, by definition, mn is odd.
Therefore, for all integers m and n, if mn is an even integer, then m is even or n is even. □

3.3, 5. Prove that the cube root of 2 is an irrational number. That is, prove that if r is a real number such that $r^3 = 2$, then r is an irrational number.

Proof.
First, as it will be needed, we will use the following lemma:

Lemma 1. $2|x^3 \Rightarrow 2|x$.

Proof. (By contrapositive) x is odd $\Rightarrow x^3$ is odd.
Let $x \in \mathbb{Z}$ be odd. Then $\exists\, k \in \mathbb{Z}$ such that $x = 2k+1$. Then

$$x^3 = (2k+1)^3 = (8k^3 + 12k^2 + 6k + 1) = 2(4k^3 + 6k^2 + 3k) + 1$$

and thus, we see that x^3 is odd.
Therefore, if $2|x^3$ then $2|x$. □

(By contradiction) Let $r \in \mathbb{R}$ such that $r^3 = 2$ and assume that r is a rational number. Then there exists $p, q \in \mathbb{Z}$ such that p and q are relatively prime and $r = \frac{p}{q}$. Then

$$r^3 = \left(\frac{p}{q}\right)^3 = 2 \Rightarrow p^3 = 2q^3$$

Thus, $2|p^3$. From above, then $2|p \Rightarrow \exists\, k \in \mathbb{Z}$ such that $p = 2k$. Thus, $p^3 = (2k)^3 = 2q^3 \Rightarrow q^3 = 4k^3$.
Thus, $4|q^3 \Rightarrow 2|q^3 \Rightarrow 2|q$.
However, this is a contradiction to our assumption that p and q were relatively prime.
Therefore, the cube root of 2 is irrational. □

3.3, 21. Evaluation of proofs.

a. The proposition is false and a counterexample is given by $m = 0$. If we exclude the possibility of having $m = 0$, the original proposition would be true, and a more appropriate proof is given below.

   Proof. (By contrapositive) For $m \in \mathbb{Z} - \{0\}$ and $x \in \mathbb{R}$, if mx is rational then x is rational.
   Let $m \in \mathbb{Z}$ and $x \in \mathbb{R}$ such that mx is rational. Then there exists integers p and q such that

$$mx = \frac{p}{q} \Rightarrow x = \frac{p}{mq}$$

   Since $m \in \mathbb{Z}$, we see that x is rational.
   Thus, if x is irrational and m is an integer, mx is irrational. □

b. The proposition is true, and the proof is correct.