

Example exercises

Amplitude modulator

$$E(t) \quad e^{-i\omega_0 t} \rightarrow A(t) = \cos \omega_0 t \rightarrow \cos \omega_0 t \cdot e^{-i\omega_0 t}$$

\downarrow

$E(w)$  

if $A(t) = \cos^2 \omega_0 t \rightarrow$ 

Phase modulator

modulation on the refractive index $n(t) = n_0 (1 + A \cos \omega_0 t)$
if $A \ll 1$, what is output spectrum? (1st order)

$$E_{out}(t) = E_0 e^{-i\omega_0 t} e^{ik_0 z \cdot n(t)} = E_0 e^{-i(k_0 z - \omega_0 t)} e^{ik_0 z A \cos \omega_0 t}$$

for $k_0 z \ll 1$ (small phase shifts)

$$E_{out}(t) \approx E_{out}(t, A=0) (1 + ik_0 z A \cos \omega_0 t)$$

$$\tilde{E}_{out}(w) = \tilde{E}_{in}(w) + ik_0 z \frac{A}{2} (\tilde{E}_{in}(w-w_1) + \tilde{E}_{in}(w+w_1))$$

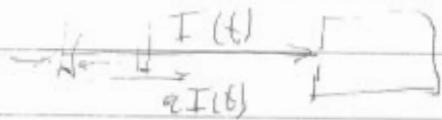
if $E_{in}(w) = E_0 \delta(w-w_0) \rightarrow$ 
cross terms in $|E|^2$ are zero.

$$\text{to second order } E_{out}(t) = E_{out}(A=0) (1 + ik_0 z A \cos \omega_0 t - (k_0 z A)^2 \cos^2 \omega_0 t)$$

$$\rightarrow$$
 

Examples

spectrometer measures $|\tilde{E}(\omega)|^2$



$$E_{int}(t) = E(t) + \sqrt{a} E(t - t_0)$$

$$\tilde{E}_{int}(\omega) = \tilde{E}(\omega) + \sqrt{a} e^{-i\omega t_0} \tilde{E}(\omega)$$

$$|\tilde{E}_{int}(\omega)|^2 = |\tilde{E}(\omega)|^2 (1 + \sqrt{a} e^{-2i\omega t_0})^2$$

$$= |\tilde{E}(\omega)|^2 (1 + \sqrt{a} (e^{i\omega t_0} + e^{-i\omega t_0}))^2$$

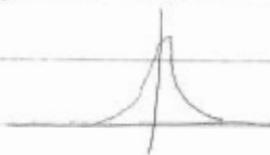
$$= |\tilde{E}(\omega)|^2 (1 + a + 2\sqrt{a} \cos \omega t_0)$$



Modulation

Suppose $E(t)$ is a signal e.g. audio

BW ≈ 20 kHz



modulate w/ $\cos \omega_0 t$

$$\text{spectrum? } \mathcal{F}(E(t) \cos \omega_0 t) = \frac{1}{2} \int E(t) e^{-i(\omega+\omega_0)t} dt + \frac{1}{2} \int E(t) e^{-i(\omega-\omega_0)t} dt \\ = \frac{1}{2} \left(\tilde{E}(\omega + \omega_0) + \tilde{E}(\omega - \omega_0) \right)$$



Mode-locked pulse train

Start in frequency space:

assume a Gaussian gain bandwidth
cavity \rightarrow comb ($\omega/\Delta\omega_c$) (ideal)

$$-(\omega - \omega_0)^2 / \Delta\omega_c^2$$

spectrum: $F(\omega) = (e^{-(\omega - \omega_0)^2 / (\Delta\omega_c^2)}) \cdot \text{comb}(\frac{\omega}{\Delta\omega_c})$
 $= G(\omega) \cdot H(\omega)$

in time domain:

$$f(t) = g(t) \otimes h(t)$$

$$g(t) = \int_{-\infty}^t e^{-i\omega_0 t} e^{-\frac{(t-t')^2}{4\Delta\omega_c^2}} dt' = e^{-i\omega_0 t} \int_{-\infty}^t e^{-\frac{(t-t')^2}{4\Delta\omega_c^2}} dt'$$

$$= \frac{1}{\sqrt{\pi t_p^2}} e^{-i\omega_0 t} e^{-\frac{t^2}{t_p^2}} \quad t_p = 2/\Delta\omega_c$$

$$h(t) = \int_{-\infty}^t \left\{ \text{comb} \left(\frac{\omega}{\Delta\omega_c} \right) \right\} dt'$$

$$\text{since } \int_{-\infty}^t \text{comb} \left(\frac{t-t'}{t_0} \right) dt' = \left(\frac{2\pi}{t_0} \right) \text{comb} \left(\frac{\omega}{2\pi/t_0} \right)$$

let $\Delta\omega_c = \pi/t_0 \quad \Delta\omega_c = 1/t_0 \quad t_0 = \text{roundtrip time}$
 $\rightarrow h(t) = \frac{1}{\Delta\omega_c} \text{comb} \left(\frac{t}{t_0} \right)$

$$F(t) = g(t) \otimes h(t) = \frac{1}{\sqrt{4\pi}} \frac{\Delta\omega_c}{\Delta\omega_c} \left(e^{-i\omega_0 t} e^{-\frac{t^2}{t_p^2}} \otimes \text{comb} \left(\frac{t}{t_0} \right) \right)$$

Linear systems

$$\xrightarrow{F(t)} \boxed{S} \xrightarrow{g(t)}$$

$$g(t) = S F(t) \quad S = \text{operator for the system}$$

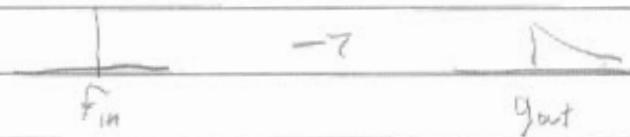
linear if $\hat{S}(a_1 f_1(t) + a_2 f_2(t)) = a_1 \hat{S}(f_1) + a_2 \hat{S}(f_2)$
 i.e. superposition holds

shift invariant if $\hat{S}(f(t-t_0)) = g(t-t_0)$
 in the time domain \hat{S} is time independent.
 spatial " \hat{S} is the same for all \vec{r}

LSI system is both

Causality:

- in time domain what happens at time t not influenced by later times $> t$



- spatial domain - different

e.g. forward propagation z is like t



$>$ system past can influence present resonance, delayed response

Impulse response

δ -function input $\delta(t) = \text{ultrashort pulse}$
 $\delta(x, y) = \text{point source.}$

$h(t) = \int \delta(t) = \text{impulse response}$

$h(t-t_0) = \int \delta(t-t_0)$ same fcn $h(t)$ if shift invariant

impulse response contains all info on LSI systems.

Transfer functions

suppose $f(t) = e^{-i\omega t}$

$$g(t) = \int f(t) = A e^{i\phi} e^{-i\omega t}$$
$$= H(\omega_0) f(t)$$

in words $e^{-i\omega t}$ = eigenfunction

$H(\omega_0)$ = eigenvalue (complex)

$A(\omega_0)$ = amplitude factor (gain or loss)

$\phi(\omega_0)$ = phase shift

A, ϕ are real functions.

$H(\omega)$ = transfer function

General input

$$g(t) = \hat{\mathcal{L}} f(t) = \hat{\mathcal{L}} \left\{ f(t') \delta(t-t') dt' \right\}$$

$\hat{\mathcal{L}}$ operates on fractions of t , not t'

$$\begin{aligned} \therefore g(t) &= \left\{ f(t') \left(\hat{\mathcal{L}} \delta(t-t') \right) dt' \right\} \\ &= \left\{ f(t') h(t-t') dt' \right\} \\ &= f \otimes h \end{aligned}$$

* convolve input w/ impulse response to get output.

in frequency space,

$$G(w) = \mathcal{F}\{f \otimes h\} = H(w)F(w)$$

↳ transfer function,

implications

systems characterization:

1) input w → measure $A(w), \phi(w)$

→ $H(w)$ → predict output $g(t)$ for any $f(t)$

2) input $\delta(t)$ measure output $h(t)$ impulse response

$$\rightarrow H(w) = \mathcal{F}\{h(t)\}$$

applications: filters, signal processing, image processing

phase effects, control in ultrafast pulses

Time-domain method:

1. calculate impulse response for etalon: what is the output if we have a delta-function input?
The output of the system in the frequency domain is the product of the spectrum of the input with the “transfer function” of the system.

$$H(\omega) = \frac{1 - r^2}{1 - r^2 \exp[-i\delta]}, \quad \delta = k_0 \cdot 2nd = \frac{2nd}{c} \omega = \omega\tau$$

The thickness and index of refraction of the etalon is d and n , respectively. The round trip transit time through the etalon is τ .

Here the spectrum of the delta-function input is constant across all frequencies, so we can get the impulse response by taking the inverse Fourier transform of $H(\omega)$:

$$h(t) = \frac{1}{2\pi} \int H(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int \frac{1 - r^2}{1 - r^2 \exp[-i\omega\tau]} e^{i\omega t} d\omega$$

To do this transform, we can write the integrand as a series (recall that's how it was derived in the first place):

$$\begin{aligned} H(\omega) &= \frac{1 - r^2}{1 - r^2 \exp[-i\omega\tau]} = (1 - r^2) \left[1 + r^2 \exp[-i\omega\tau] + (r^2 \exp[-i\omega\tau])^2 + \dots \right] = (1 - r^2) \sum_{m=0}^{\infty} r^{2m} \exp[-im\omega\tau] \\ h(t) &= \frac{(1 - r^2)}{2\pi} \int \sum_{m=0}^{\infty} r^{2m} \exp[-im\omega\tau] e^{i\omega t} d\omega \end{aligned}$$

We can exchange the order of summing and integration, and do the integral, a series of delta functions:

$$h(t) = \frac{(1 - r^2)}{2\pi} \sum_{m=0}^{\infty} r^{2m} \int \exp[i\omega(t - m\tau)] d\omega = (1 - r^2) \sum_{m=0}^{\infty} r^{2m} \delta(t - m\tau)$$

The first in the series is the transmission of the input delta function, the second is another replica, delayed by τ and reduced in amplitude by r^2 , etc.

2. Next we do the convolution of the input function with the impulse response. The input is a cosine with a rectangular envelope:

$$g(t) = E_0 \text{rect}(t/t_w) \cos(\omega_0 t)$$

the convolution is

$$\begin{aligned} f(t) &= g(t) \otimes h(t) = \int_{-\infty}^{\infty} g(t') h(t - t') dt' \\ &= E_0 \int_{-t_w/2}^{t_w/2} \cos(\omega_0 t') (1 - r^2) \sum_{m=0}^{\infty} r^{2m} \delta(t - t' - m\tau) dt' \\ &= E_0 (1 - r^2) \sum_{m=0}^{\infty} r^{2m} \int_{-t_w/2}^{t_w/2} \cos(\omega_0 t') \delta(t - t' - m\tau) dt' \\ &= E_0 (1 - r^2) \sum_{m=0}^{\infty} r^{2m} \cos(\omega_0 (t - m\tau)) \end{aligned}$$

This last step carries a condition along with it: $-t_w/2 < t - m\tau < t_w/2$. For a given t and m , the delta function is either inside the range of integration, or outside.

Frequency domain method:

Here we calculate the Fourier transform of the input, multiply it by the transfer function, then inverse transform to get the time domain response.

1. Fourier transform of the input:

$$G(\omega) = \int E_0 \text{rect}(t/t_w) \cos(\omega_o t) \exp[-i\omega t] dt = \int_{-t_w/2}^{t_w/2} \cos(\omega_o t) \exp[-i\omega t] dt$$

We did this in homework recently. We can use some of the theorems to do this transform:
Convolution theorem for the product of two time-domain functions:

$$\Im\{a(t)b(t)\} = \frac{1}{2\pi} \int F(\omega') G(\omega - \omega') d\omega'$$

The 2π factor comes from doing the inverse transforms.

$$\Im\{\cos \omega_0 t\} = 2\pi \left(\frac{1}{2} \delta(\omega - \omega_0) + \frac{1}{2} \delta(\omega + \omega_0) \right)$$

$$\Im\{\text{rect}(t/t_w)\} = t_w \text{sinc}(\omega t_w / 2)$$

$$G(\omega) = \Im\{E_0 \text{rect}(t/t_w) \cos \omega_0 t\} = E_0 \frac{t_w}{2} \left(\text{sinc} \frac{(\omega - \omega_0)t_w}{2} + \text{sinc} \frac{(\omega + \omega_0)t_w}{2} \right)$$

2. Multiply the two spectral-domain functions and do the inverse transform:

$$f(t) = \Im^{-1}\{G(\omega)H(\omega)\} = \frac{1}{2\pi} \int \left[\frac{t_w}{2} \left(\text{sinc} \frac{(\omega - \omega_0)t_w}{2} + \text{sinc} \frac{(\omega + \omega_0)t_w}{2} \right) \right] \frac{1 - r^2}{1 - r^2 \exp[-i\omega\tau]} \exp[i\omega t] d\omega$$

This looks pretty hairy, but we can use the series expansion of the etalon function again and apply the shift theorem:

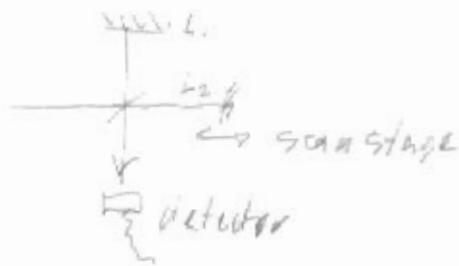
$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int \left[E_0 \frac{t_w}{2} \left(\text{sinc} \frac{(\omega - \omega_0)t_w}{2} + \text{sinc} \frac{(\omega + \omega_0)t_w}{2} \right) \right] (1 - r^2) \sum_{m=0} r^{2m} \exp[-im\omega\tau] \exp[i\omega t] d\omega \\ &= (1 - r^2) \sum_{m=0} r^{2m} \Im^{-1}\{G(\omega) \exp[-im\omega\tau]\} \\ &= (1 - r^2) \sum_{m=0} r^{2m} g(t - m\tau) \\ &= E_0 (1 - r^2) \sum_{m=0} r^{2m} \text{rect}\left(\frac{t - m\tau}{t_w}\right) \cos(\omega_0(t - m\tau)) \end{aligned}$$

Note this is the same result: if $t - m\tau$ is between $\pm t_w/2$, we add to the sum, otherwise we get zero.

Generally, convolution is more difficult than multiplication, then doing a fourier transform.

Measuring temporal coherence - field autocorrelation

Michelson



for monochromatic input

$$I_{\text{out}} = 2 I_0 (1 + \cos k(L_z - L_i))$$

no limit to extent of fringes.

for quasi monochromatic input: let $E = E(t) e^{-i\omega t}$

time dep. intensity

$$I_{\text{out}}(t) = 2 I_0 + 2 \cdot \frac{c}{8\pi} \operatorname{Re} \left[E(t + \frac{2L_i}{c}) E^*(t + \frac{2L_z}{c}) \right]$$

conjugation elim. very fast component
osc. at ω_0

detector still integrates/averages over long time scales $\sim T$

$$\bar{I}_{\text{out}}(t) = \frac{1}{T} \int_{-T/2}^{T/2} I_{\text{out}}(t') dt' = 2 I_0 + \frac{2}{T} \frac{c}{8\pi} \operatorname{Re} \left\{ \int_{-\frac{T}{2}}^{\frac{T}{2}} E(t') E^*(t' + \underbrace{\frac{2}{c}(L_z - L_i)}_{\Delta t}) dt' \right\}$$

shift variable to

$$t' = t + \frac{2L_i}{c}$$

$$t + \frac{2L_z}{c} = t' + \underbrace{\frac{2}{c}(L_z - L_i)}_{\Delta t}$$

let $T \rightarrow \infty$ in integral

$$\bar{I}_{\text{out}}(t) = 2 I_0 + \underbrace{\frac{1}{T} \left\{ |E(\omega)|^2 \right\}}_{\text{hac}(\tau)}$$

Sampling = numerical approaches.

$$\text{sample n function w/ comb } \left(\frac{x}{x_s} \right) = \sum_n \delta(x - nx_s)$$

note that for normalization, I'm defining the δ -func as having an argument w/ abs. x ,

so that $\int \delta(x - nx_s) dx = 1$

$$f_s(x) = f(x) \text{ comb}\left(\frac{x}{x_s}\right) = \sum_n f(nx_s) \delta(x - nx_s)$$

transform:

$$F_s(k) = F(k) \otimes \text{comb}\left(\frac{k}{k_s}\right)$$

$$= \frac{1}{2\pi} \sum_n F(n) \delta\left(k - n \frac{i\pi}{x_s}\right)$$

$$= \frac{i}{2\pi} \sum_n F\left(k - n \frac{\pi i}{x_s}\right)$$

$\hookrightarrow k_s = \text{sampling rate}$.

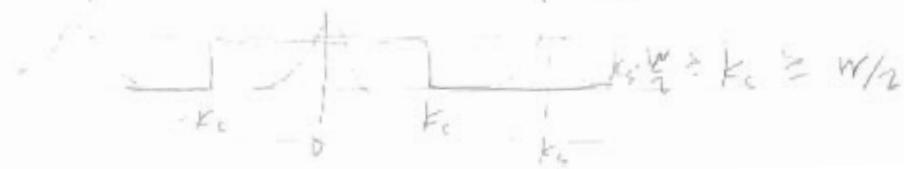
if



if no overlap in spectrum, can low-pass + recover original waveform.

> spectral full-width is W

> low-pass $H(k) = \text{rect}\left(\frac{k}{k_c}\right)$



tilted spectrum

$$G(k) = F_k(k) H(k)$$

$$= \left[\frac{1}{2\pi} \sum_n F(k - n \cdot t_s) \right] \text{rect}\left(\frac{k}{k_s}\right)$$

$$= F(k)$$

$$\text{and } g(k) = f(x)$$

Main point:

input must be bandwidth limited (full width W)
at limit $K_s = W = \frac{2\pi}{X_s}$

Sampling interval $x_s = 2\pi/W$

time domain: $W \rightarrow \Delta \omega_{\text{full}} = 2\pi f_{\text{full}}$

$$t_s = 1/f_{\text{full}}$$

max freq is $f_{\text{max}} = f_{\text{full}}/2$

$$t_s = 2/f_{\text{max}}$$

Critical sample rate = $2x$ max freq.

Under sampling \rightarrow spectrum overlap \rightarrow aliasing

typically want oversampling to allow for $W \gg W_{\text{Nyquist}}$

Note that low pass filter, $\text{rect}\left(\frac{k}{k_s}\right) \xrightarrow{\text{conv}} \text{sinc}(k_s x)$

$$f(x) \otimes \text{sinc}(k_s x) \rightarrow F(k)$$

FFT - Fast version of the discrete FT
computation time $\propto N^{\ln}$ in one dimension.

using FFT

i) sample $f(t)$ $t_s = \delta t$

Δt sets maximum bandwidth

full width $\tilde{w}_{FWH} = 2\pi/\Delta t$

so $\delta w = \frac{2\pi}{N\delta t}$

use this to determine axes in t, w space.

Normalization:

want $\sum |E_i|^2 \delta t = \sum |\tilde{E}_i|^2 \delta w$

FFT on its own pulse energy, we define
that $\sum |f_{n,i}|^2 = \sum |F_{n,i}|^2$

we must define pulse energy as

$$\mathcal{E} = \frac{1}{2} \epsilon_0 c n \sum |E_i|^2 \delta t$$

This way \mathcal{E} is independent of sampling rate.

In w domain, we want

$$\mathcal{E} = \frac{1}{2} \epsilon_0 c n \sum |\tilde{E}_i|^2 \delta w$$

∴ define $\tilde{E} = \left(\frac{\delta t}{\delta w}\right)^{1/2} \text{Fourier}[E]$

$$\left(\frac{N\delta t^2}{2\pi}\right)^{1/2}$$