MATH348: SPRING 2012 - HOMEWORK 2

INTEGRATION, ORTHOGONALITY AND FOURIER SERIES

Yes the lantern burn, burn that easy and broadcast, so raw and neatly. Thunder roll, sunshine, work it out.

Date: January 26, 2012.

Abstract. It is known that the Sturm-Liouville Problem (SLP),

$$L[y] = \lambda y, \quad l_1 y(0) + l_2 y'(0) = 0, \quad r_1 y(L) + r_2 y'(L) = 0. \tag{1}$$

where $L[y] = \dfrac{1}{w(x)}\left(-\dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right] + q(x)y\right)$, generates infinitely-many eigenfunction/eigenvalue pairs,[1] which are mutually orthogonal with respect to the inner-product

$$\langle f, g\rangle = \int_0^L f(x)\overline{g(x)}w(x)dx \tag{2}$$

where $\overline{g(x)}$ is the complex conjugate of $g(x)$. [2] Since these eigenfunctions are orthogonal, they are naturally linearly independent and it makes sense to ask what space these functions are a basis for. While this question, and its answer, is complicated and a matter of *functional analysis*, we can still see aspects of it through Fourier series. Specifically, we will form arbitrary linear combinations of the nontrivial solutions to the *periodic* SLP,

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L). \tag{3}$$

That is, we will write down functions of the form[3]

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \tag{4}$$

and ask the following questions:

1. What properties does the function f possess? That is, what can we expect of functions constructed by linear combinations of the sinusoids?
2. What are the coefficients/weights in the linear combination and more importantly, what do they mean?
3. What do the sinusoid *vectors* in the linear combination represent and how does the linear combination work so that f is constructed through the sinusoids?

What we are really doing here is studying the well-known well-celebrated Fourier series. We begin with the following problems whose purpose is explained below:

P1. It turns out, in contrast to Taylor series, that Fourier series is based on integration and because of this it is worthwhile to sharpen up our integration skills. In this problem we review standard techniques of substitution and integration by parts. Also, we practice integration of 'Dirac functions' and normal/Gaussian functions. Lastly, we show some standard orthogonality results. All of these methods and topics will come up during our study of Fourier series and later Fourier transforms.

P2. A Fourier series is this 'thing' that takes in the data from a *reasonable* function defined on a finite portion of the real line and repeats it throughout the whole plane. In other words, a Fourier series starts of with a single cell of information and makes periodic lattice/crystal from it. To do this the Fourier series begins with periodic functions and through constructive and destructive interference of waves builds the lattice/crystal in a manner consistent with the input data. Since it is important to contextualize our concepts I have provided some wikipedia readings giving more texture to the Fourier series.

P3. In physics and calculus you dealt the *vector spaces* $\mathbb{R}^2$ and $\mathbb{R}^3$.[4] A fundamental concept were standard basis vectors, $c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}} = \mathbf{y} \in \mathbb{R}^3$, where $c_1, c_2, c_3 \in \mathbb{R}$. Another example is the phase space defined by $my'' + ky = 0$ whose basis vectors are $\sin(\omega t)$ and $\cos(\omega t)$. The primary idea is that you can get to any point in the vector space by linear combinations of some set of linearly independent basis vectors and the question now is what the coefficient structure is. For $\mathbb{R}^3$ the answer is easy b/c the standard basis vectors are orthogonal. In this problem we review this concept and attempt to extend the concept of dot-product and orthogonality to functions.

P4-P5. At this point we have the construct of Fourier series built, now it's time to use them. These problems ask you to graph a periodic function, find its Fourier coefficients – and thus its Fourier series – and from this the graph of Fourier partial sums. There is a online graphing utility referenced but I will also post information on how to get Mathematica, for free, on your personal machines.

[1]In fact, more than this is known. The following are two other important results associated with SLP.

i. SLP eigenvalues are real, countable, ordered and unbounded, $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \to \infty$ as $n \to \infty$.
ii. SLP eigenfuctions have zeros/roots. Specifically, y_n associated with λ_n has $(n-1)-$many zeros/roots on $(0, L)$.

[2]An inner-product is a generalization of a dot/scalar-product. Recall from your calculus/physics courses that a dot-product is a product between two vectors that relates to their length and also the angle between them. Specifically, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ then $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$ and $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos(\theta)$ where $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ and θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. It is not hard to show that this product obeys three properties:

i. Symmetry: $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
ii. Linearity: $\mathbf{x} \cdot (\alpha\mathbf{y} + \mathbf{z}) = \alpha\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ where $\alpha \in \mathbb{R}$
iii. Positive definiteness: $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0 \iff \mathbf{x} = \mathbf{0}$

The point is that any generalization of a dot-product should also obey these there rules. If we consider the functions f and g our *vectors* then the integral given by $\langle f, g\rangle$ above returns a scalar(an area under a curve), is linear(the integral of a sum is the sum of integrals), is non-negative for $\langle f, f\rangle$(this would be the integral of a non-negative function) and zero only when f is zero. Thus, this integral can be thought of as an inner-product generalization of the dot-product.

[3]Recall the solutions to the ODE are:

$$y_1(x) = c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x), \quad c_1, c_2 \in \mathbb{R}, \lambda > 0, \tag{5}$$

$$y_2(x) = c_3 \sinh(\sqrt{|\lambda|}x) + c_4 \cosh(\sqrt{|\lambda|}x), \quad c_3, c_4 \in \mathbb{R}, \lambda < 0, \tag{6}$$

$$y_3(x) = c_5 x + c_6, \quad c_5, c_6 \in \mathbb{R}, \lambda = 0. \tag{7}$$

Of these solutions, the only ones that are periodic are those associated with c_1, c_2 and c_6.

[4]Someone once told me that a *vector space* is a house where you keep your vectors. This isn't far from the truth. Even if you haven't been formally exposed to a vector space, say in a linear algebra class, you have been using vector spaces tacitly for years. The basic idea is that a vector space is a collection of mathematical objects that behave like vectors from $\mathbb{R}^3$. That is, if you create a linear combination of these vectors you get yet another vector from the space. Seems dull, and sometimes math is dull, but what is gained is that ANY mathematical object that obeys the rules you know about $\mathbb{R}^3$ can be called a vector. So, from this perspective $\mathbb{R}^2$ is really not that much different than the simple harmonic oscillator phase space. From this perspective the questions become:

Q1. How do you systematically specify points in a vector space?
Q2. How do you find the coefficients of the linear combination?
Q3. Is there a geometric structure to the vector space?

We have answers to these questions:

A1. A vector space has a basis, which is a collection of linearly independent vectors whose size is equal to the number of degrees of freedom or dimension of the vector space. An point in the vector space can be specified by a linear combination of vectors from the basis.
A2. If we specify a point in the space through a linear combination then it is natural to ask what the coefficients in this specific linear combination are – another way of saying this is what are the coordinates of the point relative to the basis elements. If the number of basis elements is finite then the problem is a standard problem from linear algebra. If, however, the number of basis elements is infinite then any iterative scheme we can think of is destined to fail.
A3. The geometric structure of $\mathbb{R}^n$ is given through the dot-product. Often, for functions, it is possible to generalize the dot-product to something called an inner-product and from this define the notion of angle and length.

1. Integration Review

We will see, because of mathematical abstraction of the scalar-product on vectors, that integration is the primary tool of Fourier analysis. To prepare, we consider the following integrations.

1.1. Integration by Parts. $\int x^3 \cos(5x)dx$

1.2. Integration by ? $\int x^2 \sin(2x^3)dx$ some $x_0 \in \mathbb{R}$

1.3. Orthogonality. Show that $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0$ for all $n, m \in \mathbb{N}$

1.4. More Orthogonality. Show that $\int_a^b e^{i\frac{n\pi}{L}x} e^{-i\frac{m\pi}{L}x} dx = 2L\delta_{mn}$ where $L = \frac{b-a}{2}$ and for all $n, m \in \mathbb{Z}$.[5]

1.5. Tricky IBP or Tricky Algebra. $\int e^{ax}\cos(bx)dx$ and $\int e^{ax}\sin(bx)dx$

1.6. Integration of *Delta 'Functions'*. Justify that $\int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx = g(x_0)$ for $x_0 \in \mathbb{R}$. [6]

1.7. Integrals of *Gaussian Functions*. Show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

2. Introduction to Fourier Series

2.1. Wikipedia. Go to `http://en.wikipedia.org/wiki/Fourier_series` and read the introductory material on Fourier Series and describe in your own words the purpose and application of Fourier Series.

2.2. Graphing. Using the Java Applet found at `http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/Fourier/fourier.html`, use the applet to graph a truncated Fourier Series approximating the saw-tooth function. What occurs at the points jump-discontinuity?

---

[5] Here the function δ_{mn} is called Kronecker delta function, `http://en.wikipedia.org/wiki/Kronecker_delta`, and is formally defined as

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} . \tag{8}$$

[6] Here the δ is the so-called Dirac, or continuous, delta function. This isn't a function in the true sense of the term but instead what is called a *generalized function*. For more information on this matter consider `http://en.wikipedia.org/wiki/Dirac_delta_function`. To drive home that this *function* is strange, let me spoil the punch-line. The sampling function $f(x) = \text{sinc}(ax)$ can be used as a definition for the Delta *function* as $a \to 0$. So can a bell-curve probability distribution. Yikes!

2.3. **Truncated Fourier Series.** Read, as much as you can, of `http://en.wikipedia.org/wiki/Gibbs_phenomenon`. The sum of a finite, or infinite amount of periodic functions is periodic. Is this always true for both finite and infinite sums of continuous functions? Can you think of a counterexample? [7]

3. Inner-products and Orthogonality

Given,

$$\hat{\mathbf{i}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \hat{\mathbf{j}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{9}$$

3.1. **Orthonormality.** Show that the vectors are orthonormal by verifying the inner-products $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1$.

3.2. **Orthogonal Representation I.** Show that any vector for $\mathbb{R}^2$ can be created as a linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$. That is, given,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}, \tag{10}$$

find c_1, c_2, can be found in terms of x_1 and x_2.

3.3. **Orthogonal Representation II.** Based on the derivation from class and problem 1.4 from this homework we know that for the inner-product,

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx \tag{11}$$

were $2L = b - a$ we have

$$\langle \cos(\omega_n x), \cos(\omega_m x)\rangle = \langle \sin(\omega_n x), \sin(\omega_m x)\rangle = L\delta_{nm}, \tag{12}$$

$$\langle \cos(\omega_n x), \sin(\omega_m x)\rangle = 0 \tag{13}$$

---

[7] These questions are meant to lead you. Remembering that sine and cosine are examples of continuous periodic functions, you should be thinking about the following string of thoughts.

i. Fourier series represent an 'arbitrary' periodic function in terms of known periodic functions.
ii. Increasing the number of terms in a Fourier series creates better and better sinusoidal waveform fits of the function f and in the limit of infinitely many terms this fit is exact 'almost-everywhere'.
iii. Hopefully by the time you do this problem we would have mentioned in class that the Fourier series representation of a function converges in the sense of averages and that since jump-discontinuities are integrable-discontinuities the Fourier series would average the right and left hand limits of the function at the point of discontinuity. This will happen indifferent to the actual value of the function at the point of discontinuity. Thus the Fourier series may actually differ from its function at the boundaries of its periodic-domains! In this way we take = to mean equality *almost everywhere* (`http://en.wikipedia.org/wiki/Almost_everywhere`).

So, we have that the sawtooth example from class and the square-wave example online are examples where the infinite sum of continuous periodic functions converges to a periodic function with jump-discontinuities.

for $n, m \in \mathbb{N}$ and $\omega_n = n\pi/L$. Show that for $b = -a = \pi$ the coefficients in the linear combination $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ are

$$a_0 = \frac{1}{2\pi} \langle f(x), 1 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \tag{14}$$

$$a_n = \frac{1}{\pi} \langle f(x), \cos(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \tag{15}$$

$$b_n = \frac{1}{\pi} \langle f(x), \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx. \tag{16}$$

4. Fourier Series for an Even 2π−periodic Function

Let $f(x) = x^2$ for $x \in (-\pi, \pi)$ be such that $f(x + 2\pi) = f(x)$.

4.1. **Graphing.** Sketch f on $(-2\pi, 2\pi)$.

4.2. **Symmetry.** Is the function even, odd or neither?

4.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

4.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f.

5. Fourier Series for an Oddish 2π−periodic Function

Let $f(x) = x + \alpha$ for $x \in (-\pi, \pi)$ and $\alpha \in \mathbb{R}$ be such that $f(x + 2\pi) = f(x)$.

5.1. **Graphing.** Sketch f on $(-2\pi, 2\pi)$.

5.2. **Symmetry.** Is the function even, odd or neither?

5.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

5.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f assuming that $\alpha = 1$.

(Scott Strong) Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401

E-mail address: `sstrong@mines.edu`