# EM waves: energy, resonators

Scalar wave equation

Maxwell equations to the EM wave equation

A simple linear resonator

Energy in EM waves

3D waves

# Simple scalar wave equation

- $2^{nd}$ order PDE $$\frac{\partial^2}{\partial z^2}\psi(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(z,t) = 0$$
- Assume separable solution $$\psi(z,t) = f(z)g(t)$$

$$\frac{1}{f(z)}\frac{\partial^2}{\partial z^2}f(z) - \frac{1}{c^2}\frac{1}{g(t)}\frac{\partial^2}{\partial t^2}g(t) = 0$$

  – Each part is equal to a constant *A*

$$\frac{1}{f(z)}\frac{\partial^2}{\partial z^2}f(z) = A,\ \frac{1}{c^2}\frac{1}{g(t)}\frac{\partial^2}{\partial t^2}g(t) = A$$

$$f(z) = \cos(kz) \rightarrow -k^2 = A,\ g(t) = \cos(\omega t) \rightarrow -\omega^2\frac{1}{c^2} = A$$

$$\omega = \pm k\,c$$

Sin( ) also works as a second solution

# Full solution of wave equation

- Full solution is a linear combination of both

$$\psi(z,t) = f(z)g(t) = \left(A_1 \cos kz + A_2 \sin kz\right)\left(B_1 \cos \omega t + B_2 \sin \omega t\right)$$

- Too messy: use complex solution instead:

$$\psi(z,t) = f(z)g(t) = \left(A_1 e^{ikz} + A_2 e^{-ikz}\right)\left(B_1 e^{i\omega t} + B_2 e^{-i\omega t}\right)$$

$$\psi(z,t) = A_1 B_1 e^{i(kz+\omega t)} + A_2 B_2 e^{-i(kz+\omega t)} + A_1 B_2 e^{i(kz-\omega t)} + A_2 B_1 e^{-i(kz-\omega t)}$$

  - Constants are arbitrary: rewrite

$$\psi(z,t) = A_1 \cos\left(kz + \omega t + \phi_1\right) + A_2 \cos\left(kz - \omega t + \phi_2\right)$$

# Interpretation of solutions

- Wave vector $k = \frac{2\pi}{\lambda}$
- Angular frequency $\omega = 2\pi\nu$
- Wave total phase: $\Phi = kz - \omega t + \phi$
  - "absolute phase": $\phi$
  - Phase velocity: *c* $\Phi = kz - k\,c\,t + \phi = k\left(z - c\,t\right) + \phi$

$\Phi = \text{constant when } z = ct$

$$\psi(z,t) = A_1 \cos\left(kz + \omega t + \phi_1\right) + A_2 \cos\left(kz - \omega t + \phi_2\right)$$

Reverse (to -z) Forward (to +z)

# Maxwell's Equations to wave eqn

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla} \cdot \mathbf{E} = 0 \qquad \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0 \qquad \vec{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}$$

Take the curl:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = -\frac{\partial}{\partial t}\vec{\nabla} \times \mathbf{B} = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}\right)$$

Use the vector ID:

$$\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right) = \mathbf{B}\left(\mathbf{A} \cdot \mathbf{C}\right) - \mathbf{C}\left(\mathbf{A} \cdot \mathbf{B}\right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \mathbf{E}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\mathbf{E} = -\vec{\nabla}^2 \mathbf{E}$$

$$\boxed{\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}}$$

"Inhomogeneous Wave Equation"

# Maxwell's Equations in a Medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- The polarization (**P**) can be thought of as the driving term for the solution to this equation, so the polarization determines which frequencies will occur.
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0\chi\mathbf{E}$$

- In nonlinear optics, the induced polarization is more complicated:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0\left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E}^2 + \chi^{(3)}\mathbf{E}^3 + \ldots\right)$$

- The extra nonlinear terms can lead to new frequencies.

# Solving the wave equation: linear induced polarization

For low irradiances, the polarization is proportional to the incident field:

$$\mathbf{P}\left(\mathbf{E}\right) = \varepsilon_0 \chi \mathbf{E}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \left(1+\chi\right)\mathbf{E} = \varepsilon \mathbf{E} = n^2 \mathbf{E}$$

In this simple (and most common) case, the wave equation becomes:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2}\chi \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \rightarrow \vec{\nabla}^2 \mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Using: $\varepsilon_0 \mu_0 = 1/c^2$ $\qquad \varepsilon_0\left(1+\chi\right) = \varepsilon = n^2$

The electric field is a vector function in 3D, so this is actually 3 equations:

$$\vec{\nabla}^2 E_x\left(\mathbf{r},t\right) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_x\left(\mathbf{r},t\right) = 0$$

$$\vec{\nabla}^2 E_y\left(\mathbf{r},t\right) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_y\left(\mathbf{r},t\right) = 0$$

$$\vec{\nabla}^2 E_z\left(\mathbf{r},t\right) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2} E_z\left(\mathbf{r},t\right) = 0$$

# Plane wave solutions for the wave equation

If we assume the solution has no dependence on x or y:

$$\vec{\nabla}^2\mathbf{E}(z,t)=\frac{\partial^2}{\partial x^2}\mathbf{E}(z,t)+\frac{\partial^2}{\partial y^2}\mathbf{E}(z,t)+\frac{\partial^2}{\partial z^2}\mathbf{E}(z,t)=\frac{\partial^2}{\partial z^2}\mathbf{E}(z,t)$$

$$\rightarrow \frac{\partial^2\mathbf{E}}{\partial z^2}-\frac{n^2}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \quad = \quad 0$$

The solutions are oscillating functions, for example

$$\mathbf{E}(z,t)=\hat{\mathbf{x}}\,E_x\cos\left(k_z z-\omega t\right)$$

Where $\omega = k\,c, \quad k = 2\pi n/\lambda, \quad v_{ph} = c/n$

This is a linearly polarized wave.

# Complex notation for waves

- Write cosine in terms of exponential

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t + \phi) = \hat{\mathbf{x}} E_x \frac{1}{2}\left(e^{i(kz-\omega t+\phi)} + e^{-i(kz-\omega t+\phi)}\right)$$

  - Note E-field is a *real* quantity.
  - It is convenient to work with just one part
    - We will use $E_0 e^{+i(kz-\omega t)}$ $E_0 = \frac{1}{2} E_x e^{i\phi}$
    - Svelto: $e^{-i(kz-\omega t)}$
  - Then take the real part.
    - No factor of 2
    - In *nonlinear* optics, we have to explicitly include conjugate term

# Example: linear resonator (1D)

- Boundary conditions: conducting ends (mirrors)

$$E_x(z=0,t)=0 \qquad E_x(z=L_z,t)=0$$

- Field is a superposition of +'ve and –'ve waves:

$$E_x(z,t)=A_+e^{i(k_z z-\omega t+\phi_+)}+A_-e^{i(-k_z z-\omega t+\phi_-)}$$

  – Absorb phase into complex amplitude

$$E_x(z,t)=\left(A_+e^{+ik_z z}+A_-e^{-ik_z z}\right)e^{-i\omega t}$$

  – Apply b.c. at z = 0

$$E_x(0,t)=0=\left(A_++A_-\right)e^{-i\omega t}\rightarrow A_+=-A_-$$

$$E_x(z,t)=A\sin k_z z\, e^{-i\omega t}$$

# Quantization of frequency: longitudinal modes

- Apply b.c. at far end

$$E_x\left(L_z,t\right)=0=A\sin k_z L_z\, e^{-i\omega t} \qquad \rightarrow k_z L_z = l\pi \qquad l=1,2,3,\cdots$$

  – Relate to wavelength:

$$k_z=\frac{2\pi}{\lambda}=\frac{l\pi}{L_z}\rightarrow L_z=l\frac{\lambda}{2}$$

Integer number of half-wavelengths

  – Relate to allowed frequencies:

$$\frac{\omega_l}{c}=\frac{l\pi}{L_z}\rightarrow \nu_l=l\frac{c}{2L_z}$$

  – Equally spaced frequencies:

$$\Delta\nu=\frac{c}{2L_z}=\frac{1}{T_{RT}}$$

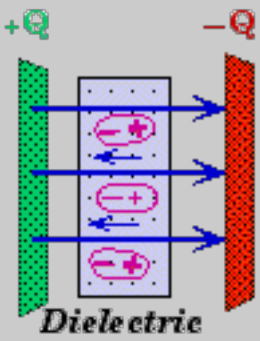
Frequency spacing = 1/ round trip time

# Wave energy and intensity

- Both E and H fields have a corresponding energy density (J/m$^3$)
  - For static fields (e.g. in capacitors) the energy density can be calculated through the work done to set up the field

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2$$

  - Some work is required to polarize the medium
  - Energy is contained in both fields, but H field can be calculated from E field



# Calculating H from E in a plane wave

- Assume a non-magnetic medium

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(kz - \omega t)$$

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

  – Can see **H** is perpendicular to **E**

$$-\mu_0 \frac{\partial \mathbf{H}}{\partial t} = \vec{\nabla} \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \hat{\mathbf{y}} \partial_z E_x = -\hat{\mathbf{y}} k_z E_0 \sin(k_z z - \omega t)$$

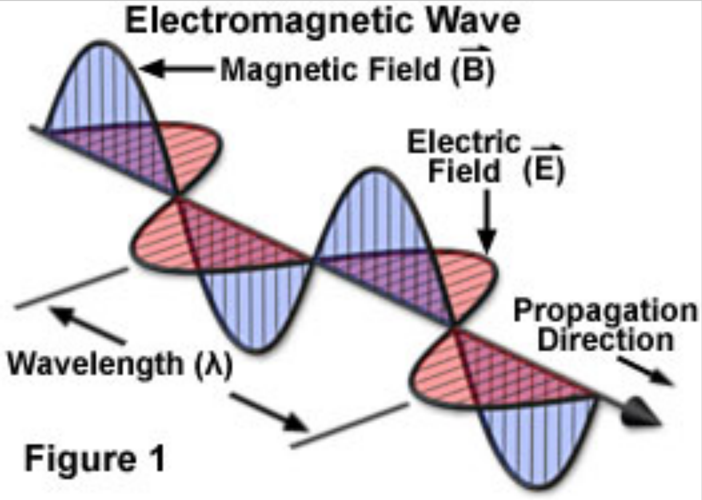
  – Integrate to get **H**-field:

$$\mathbf{H} = \hat{\mathbf{y}} \int \frac{k_z E_0}{\mu_0} \sin(k_z z - \omega t) dt = \hat{\mathbf{y}} \frac{k_z E_0}{\mu_0} \left( \frac{-\cos(k_z z - \omega t)}{-\omega} \right)$$

# H field from E field

- H field for a propagating wave is *in phase* with E-field

$$\mathbf{H} = \hat{\mathbf{y}} H_0 \cos\left(k_z z - \omega t\right)$$
$$= \hat{\mathbf{y}} \frac{k_z}{\omega \mu_0} E_0 \cos\left(k_z z - \omega t\right)$$




- Amplitudes are not independent

$$H_0 = \frac{k_z}{\omega \mu_0} E_0 \qquad k_z = n\frac{\omega}{c} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0} \rightarrow \frac{1}{\mu_0 c} = \varepsilon_0 c$$

$$H_0 = \frac{n}{c \mu_0} E_0 = n \varepsilon_0 c E_0$$

# Energy density in an EM wave

- Back to energy density, non-magnetic

$$\rho = \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu_0 H^2$$

$$H = n\varepsilon_0 c E$$

$$\varepsilon = \varepsilon_0 n^2$$

$$\rho = \tfrac{1}{2}\varepsilon_0 n^2 E^2 + \tfrac{1}{2}\mu_0 n^2 \varepsilon_0^{\ 2} c^2 E^2$$

$$\mu_0 \varepsilon_0 c^2 = 1$$

$$\rho = \varepsilon_0 n^2 E^2 = \varepsilon_0 n^2 E^2 \cos^2\left(k_z z - \omega t\right)$$

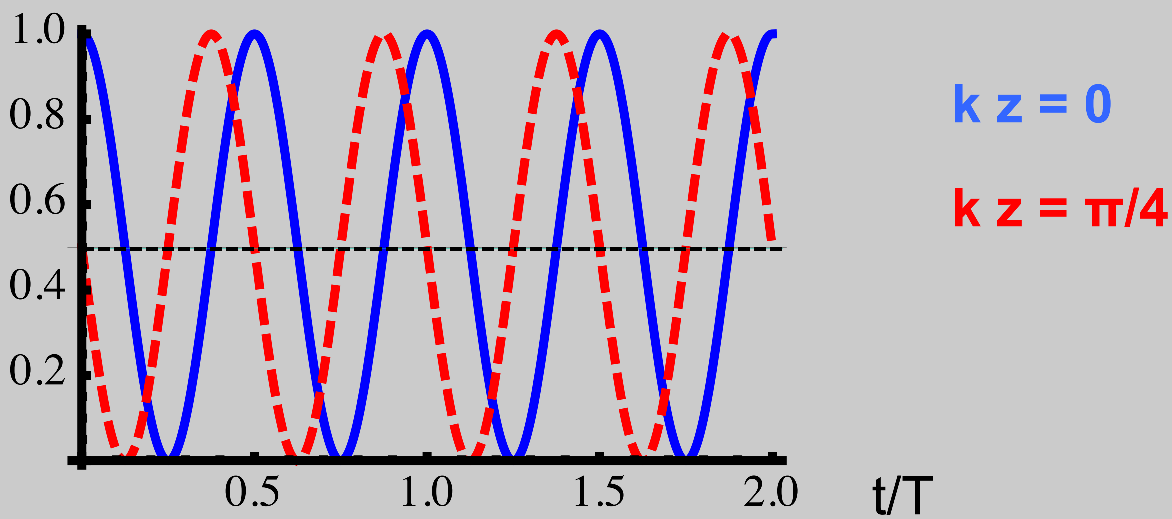
***Equal energy*** in both components of wave

# Cycle-averaged energy density

- Optical oscillations are faster than detectors
- Average over one cycle:

$$\langle \rho \rangle = \varepsilon_0 n^2 E_0{}^2 \frac{1}{T} \int_0^T \cos^2\left(k_z z - \omega t\right) dt$$

  – Graphically, we can see this should = ½




  – Regardless of position z

$$\langle \rho \rangle = \frac{1}{2} \varepsilon_0 n^2 E_0{}^2$$

# Intensity and the Poynting vector

- Intensity is an energy flux (J/s/cm$^2$)
- In EM the Poynting vector give energy flux

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

  – For our plane wave,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = E_0 \cos\left(k_z z - \omega t\right) n\varepsilon_0 c E_0 \cos\left(k_z z - \omega t\right) \hat{\mathbf{x}} \times \hat{\mathbf{y}}$$

$$\mathbf{S} = n\varepsilon_0 c E_0^2 \cos^2\left(k_z z - \omega t\right) \hat{\mathbf{z}}$$

  – **S** is along **k**

- Time average: $\mathbf{S} = \frac{1}{2} n \varepsilon_0 c E_0^2 \hat{\mathbf{z}}$
- *Intensity* is the magnitude of **S**

$$I = \frac{1}{2} n\varepsilon_0 c E_0^2 = \frac{c}{n}\rho = V_{phase} \cdot \rho$$

**Photon flux:** $F = \dfrac{I}{h\nu}$

# General 3D plane wave solution

- Assume separable function

$$\mathbf{E}(x,y,z,t) \sim f_1(x) f_2(y) f_3(z) g(t)$$

$$\vec{\nabla}^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(z,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} = \mathbf{E_0} e^{i\left(k_x x + k_y y + k_z z\right)} e^{-i\omega t}$$

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

  – Now k-vector can point in arbitrary direction

- With this solution in W.E.:

$$\boxed{n^2 \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}}$$

Valid even in waveguides and resonators

# Grad and curl of 3D plane waves

- Simple trick:

$$\nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z$$

  – For a plane wave,

$$\nabla \cdot \mathbf{E} = i\left(k_x E_x + k_y E_y + k_z E_z\right) = i\left(\mathbf{k} \cdot \mathbf{E}\right)$$

  – Similarly,

$$\nabla \times \mathbf{E} = i\left(\mathbf{k} \times \mathbf{E}\right)$$

- Consequence: since $\nabla \cdot \mathbf{E} = 0, \quad \mathbf{k} \perp \mathbf{E}$
  – For a given k direction, E lies in a plane
  – E.g. x and y linear polarization for a wave propagating in z direction

# Writing electric field expressions: 1D

- Write a complex (phasor) expression for an E-field linearly polarized in the x-direction, propagating in the z direction. Frequency ω, wavenumber k.

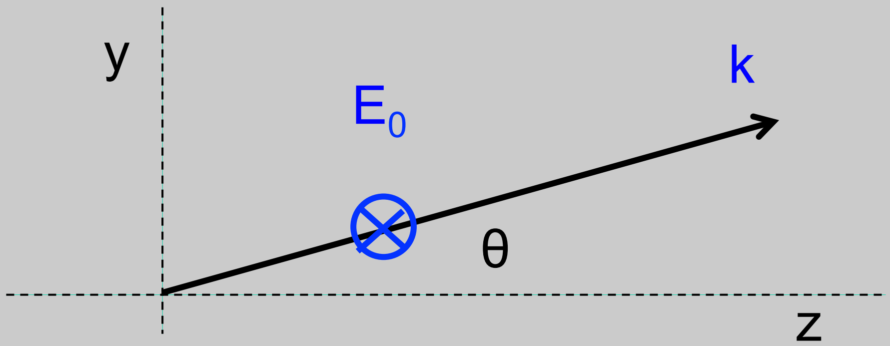
$$E(z,t) = E_0 \exp(ikz - i\omega t)$$

- Write an expression for the field of a *standing* wave (E=0 at z = 0 and z = L) and for the allowed k's.

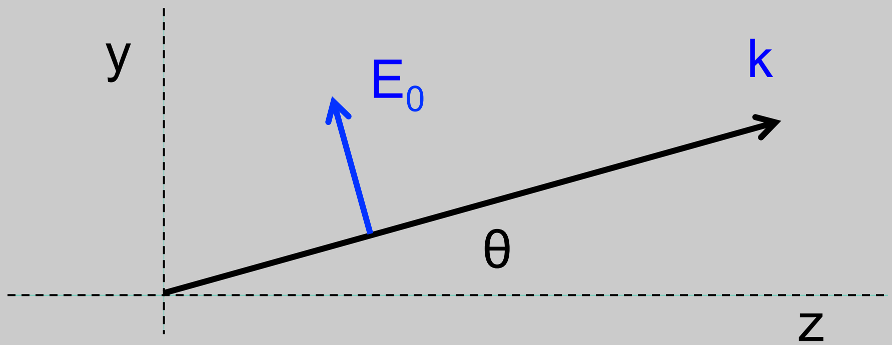
$$E(z,t) = E_0 \sin(k_n z) \exp(-i\omega t), \ \text{with } k_n = n\pi / L$$

# Writing expressions for waves: 3D

- Write an expression for a complex E-field as shown:



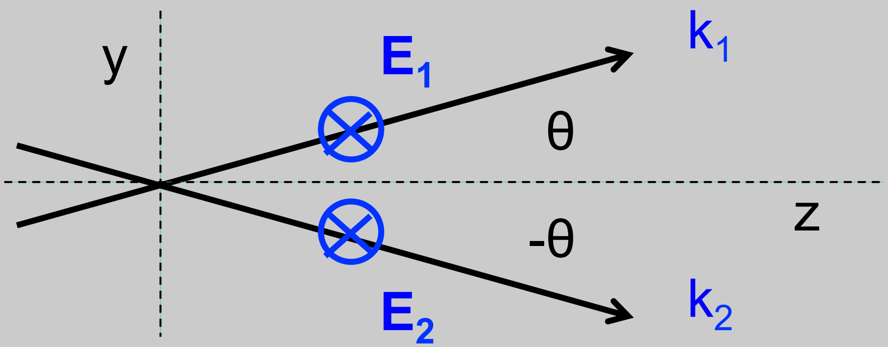

$$\mathbf{E}(y,z,t) = E_0 \hat{\mathbf{x}} e^{i(k y \sin\theta + k z \cos\theta - \omega t)}$$




$$\mathbf{E}(y,z,t) = E_0 \left[ \hat{\mathbf{y}} \cos\theta - \hat{\mathbf{z}} \sin\theta \right] e^{i(k y \sin\theta + k z \cos\theta - \omega t)}$$

# Interference in 2D

- Write an expression for the total field (sum of the fields as shown. Assume equal amplitude fields.




$$\mathbf{E}(y,z,t) = E_0 \hat{\mathbf{x}} \left( e^{+iky\sin\theta} + e^{-iky\sin\theta} \right) e^{i(kz\cos\theta - \omega t)}$$

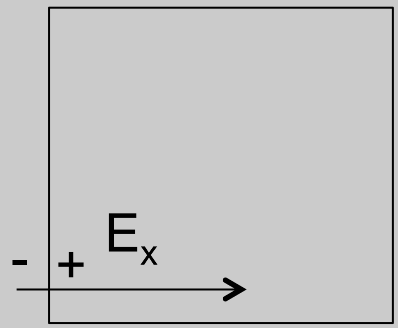
- Now write an expression for the intensity at z = 0. Just write the peak intensity as $I_0$.

$$I(y,z,t) = I_0 \cos^2(ky\sin\theta) = I_1 + I_2 + \sqrt{I_1 I_2}\cos(2ky\sin\theta)$$

# Closed box resonator: blackbody cavity

- Here we have a 3D pattern of standing waves
  - Exact boundary conditions aren't imp't, but for conducting walls:
    - E=0 where field is parallel to wall
    - Slope E=0 where field is perp to wall (charges can accumulate there)
  - Example standing wave solution:

$$E_x(x,y,z) = A_x \cos k_x x \sin k_y y \sin k_z z$$




    - Cos( ) function along field direction
  - Others:

$$E_y(x,y,z) = A_y \sin k_x x \cos k_y y \sin k_z z$$

$$E_z(x,y,z) = A_z \sin k_x x \sin k_y y \cos k_z z$$

# Discrete wavevectors

- Discrete values of k:

$$k_x = \frac{l\pi}{L_x} \qquad k_y = \frac{m\pi}{L_y} \qquad k_z = \frac{n\pi}{L_z}$$

- With these solutions in the wave equation

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}$$

2 allowed polarizations

  – k's are discrete, so there are discrete allowed frequencies:

$$\omega_{lmn} = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2}$$

$$\nu_{lmn} = \frac{c}{2\pi}\sqrt{k_x^2 + k_y^2 + k_z^2} = c\sqrt{\left(\frac{l}{2L_x}\right)^2 + \left(\frac{m}{2L_y}\right)^2 + \left(\frac{n}{2L_z}\right)^2}$$

# Field in equilibrium with walls: classical

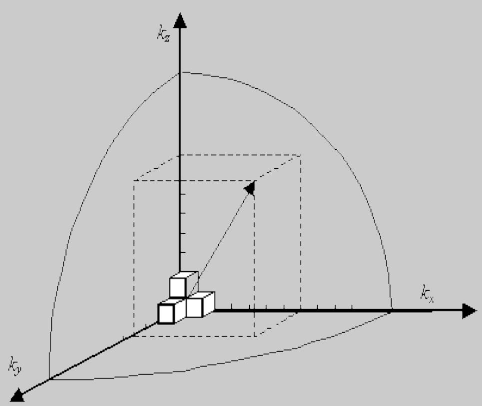
- Hold cavity walls at temperature T
- What is probability that a mode will be excited?
- Classical view (Boltzmann): $P(\mathcal{E}) \propto e^{-\mathcal{E}/kT}$
  - assume the amount of energy in each mode can take any value (continuous range) this is wrong!
  - average energy for each mode is

$$\langle \mathcal{E} \rangle = \frac{\int_0^\infty \mathcal{E}\, P(\mathcal{E})\, d\mathcal{E}}{\int_0^\infty P(\mathcal{E})\, d\mathcal{E}} = \frac{\int_0^\infty \mathcal{E}\, e^{-\mathcal{E}/kT}\, d\mathcal{E}}{\int_0^\infty e^{-\mathcal{E}/kT}\, d\mathcal{E}} = kT$$

  - Note: this is not kT/2 as in equipartition of K.E. There, integrate on velocity, which ranges – to +

# Density of states

- For a given box size, there is a low frequency cutoff but no cutoff for high frequencies
- Near a given frequency, there will be a number of combinations of k's l,m,n for that frequency




$$N\left(k\right) = \#\text{pol states} \times \frac{\text{volume of k-space octant}}{\text{volume of unit k-space cell}}$$

$$= 2\frac{1}{8}\frac{\left(4/3\right)\pi k^3}{\frac{\pi}{L_x}\times\frac{\pi}{L_y}\times\frac{\pi}{L_z}} = \frac{k^3}{3\pi^2}V$$

Density of modes = density of states $\quad g\left(k\right)dk = \frac{1}{V}\frac{dN\left(k\right)}{dk}dk = \frac{k^2}{\pi^2}dk$

Other forms: $\quad g\left(\omega\right)d\omega = \frac{\omega^2}{\pi^2 c^3}d\omega \quad g\left(\nu\right)d\nu = 8\pi\frac{\nu^2}{c^3}d\nu$

# Spectral energy density

- Generalize EM energy density to allow for spectral distribution

$$\rho(\nu)d\nu = \text{excitation energy per mode} \times \text{density of modes}$$

  - Total energy density: $\int \rho(\nu)d\nu$
  - Classical form:

$$\rho(\nu)d\nu = k_B T \, \frac{8\pi\nu^2}{c^3} d\nu$$

  - Problem: total energy is infinite!
- Planck: only allow quantized energies for each mode

$\mathcal{E} = \left(n + \tfrac{1}{2}\right)h\nu$ n = number of photons in each mode

  - Now get average energy/mode with sum, not integral

$$P_n = \frac{e^{-\mathcal{E}_n/k_B T}}{\sum_j e^{-\mathcal{E}_j/k_B T}}$$

Mean photon number: $\bar{n} = \sum_n n P_n$

# Blackbody spectrum

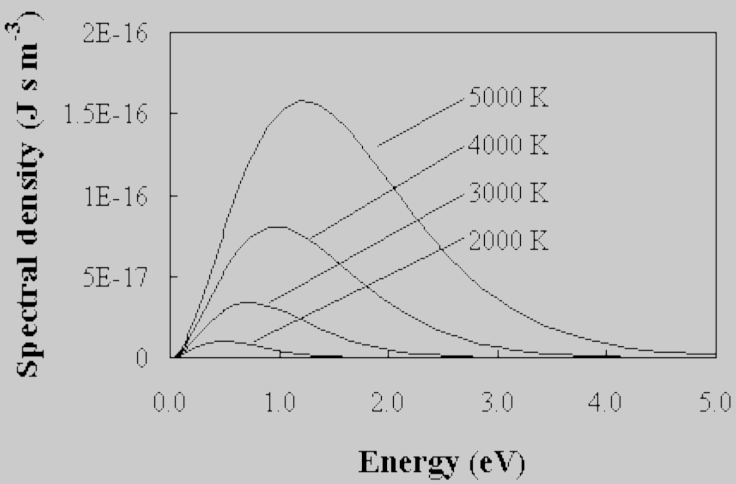
- Mean number of photons per mode:

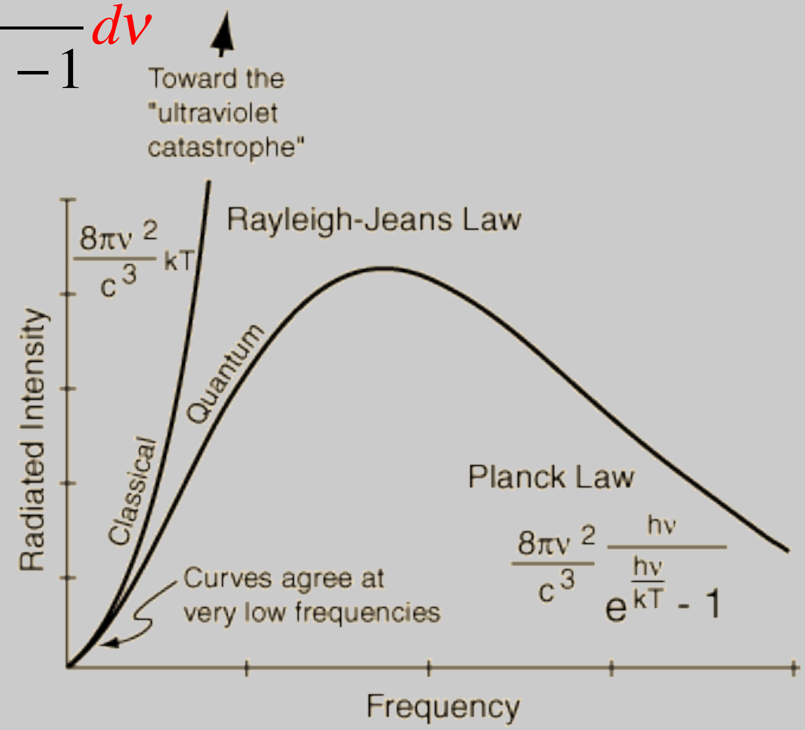
$$\bar{n} = \sum_j n P_n = 1 \Big/ \left( e^{h\nu/k_B T} - 1 \right)$$

- Spectral energy density of BB radiation:

$$\rho(\nu)d\nu = \text{avg \# photons per mode} \times {\color{blue} h\nu \text{ per photon}} \times {\color{red} \text{density of modes}}$$

$$= \frac{1}{e^{h\nu/k_B T} - 1} {\color{blue} h\nu} \, {\color{red} g(\nu) d\nu} = {\color{red} 8\pi \frac{\nu^2}{c^3}} \frac{{\color{blue} h\nu}}{e^{h\nu/k_B T} - 1} {\color{red} d\nu}$$



# Wave propagation with absorption

- Consider light absorption from a thin slab

$$I_1 = I_0 - I_0\,\alpha\,\Delta z$$

- Generalize to an equation for arbitrary length:

$$I_1 - I_0 = \Delta I = -I_0\,\alpha\,\Delta z \rightarrow \frac{dI}{dz} = -\alpha\, I$$

$$I(z) = I_0\, e^{-\alpha z}$$ Beer's Law

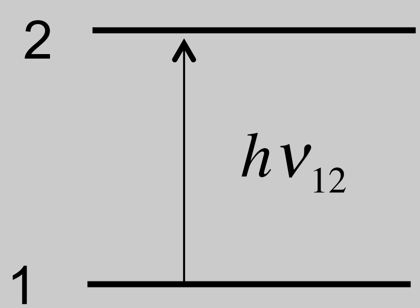
- Absorption coefficient (units $m^{-1}$) is proportional to the number density of absorbers:

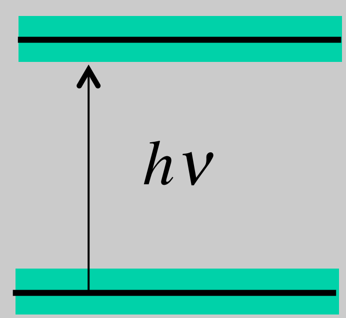
$$\alpha = N_1 \sigma$$

  – $N_1$ = number density ($m^{-3}$) of species in level 1
  – σ? Has units of $m^2$, = "cross-section"

# Models for σ: hard and soft spheres

- Consider an collection of “black” spheres that absorb if struck by a photon.
- Cross-section for absorption is just the projected area of the sphere. $\sigma = \pi a^2$
- For an atom, the probability of absorption depends on how close the incident frequency is to resonance:




Absorption lines are *broadened*, so *exact* energy is not required.

$$\sigma \rightarrow \sigma(\nu)$$

# Example: absorption of pump light in

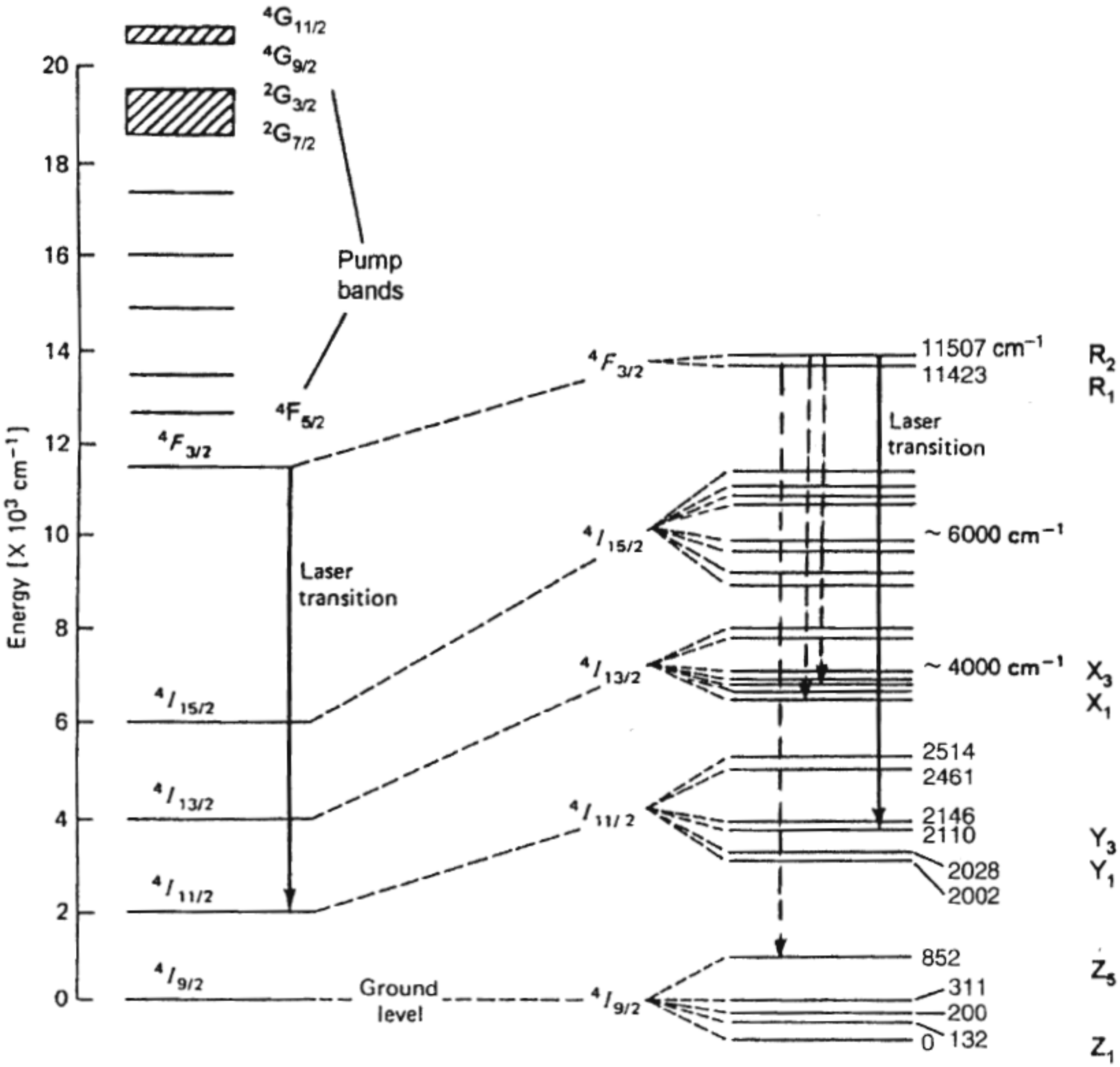
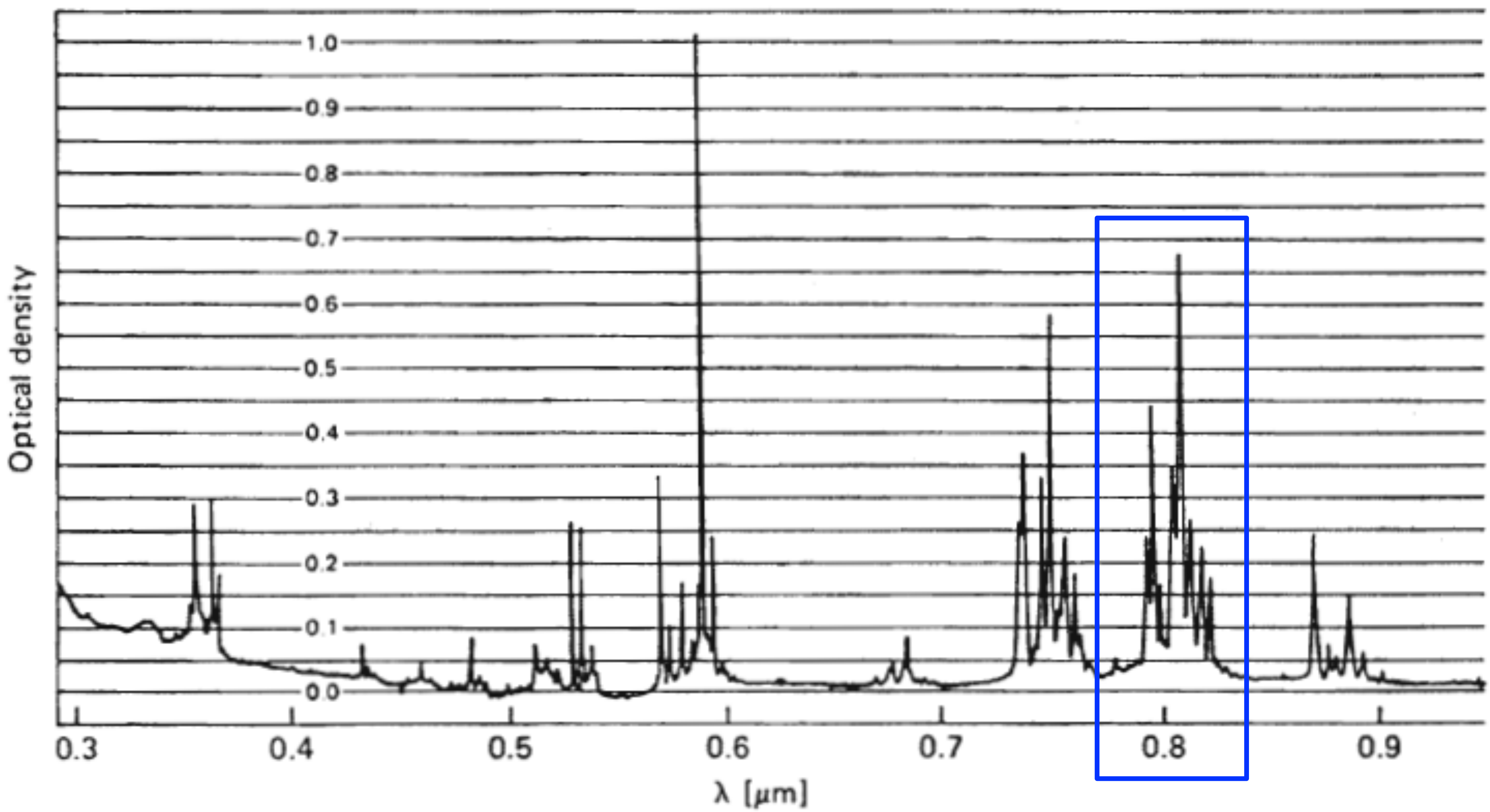



**Fig. 2.2.** Energy level diagram of Nd:YAG. The solid line represents the major transition at 1064 nm, and the dashed lines are the transitions at 1319, 1338, and 946 nm.

- $Nd^{3+}$ is a heavy ion with many possible transitions
- Pump to anywhere above the $^4F_{3/2}$ level

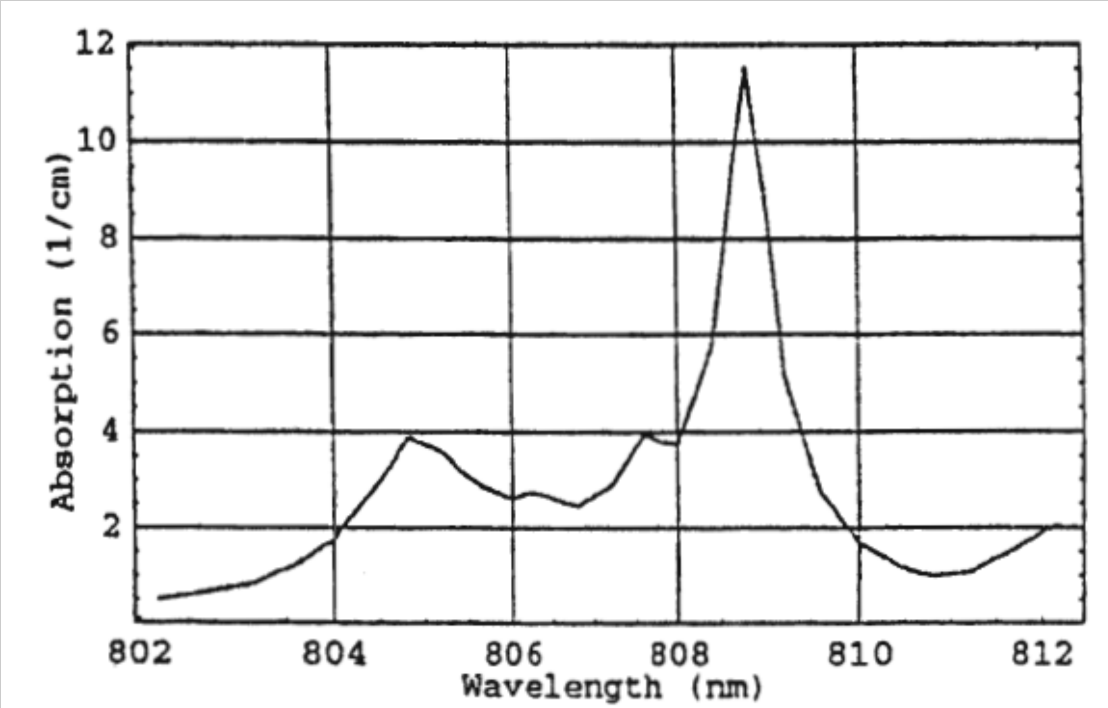
# Absorption spectrum of $Nd^{3+}$:YAG




- Optical density (OD) = $-\log_{10}[T]$

# Pump bands near 808nm

- Powerful laser diodes (LD) are available near 808nm




3mm thick Nd:YAG crystal

- What % is absorbed at the peak (α=11/cm)?
- What is the OD?
- If $N_{Nd}=1.38\times10^{20}/cm^3$ (1% atomic), what is the absorption cross-section?

- Note: LD output wavelength depends on temperature, so this needs tuning and stabilization in real systems.

# Transition rates

- We have been looking from of the point of view of the photons. What about the atoms?
  - Absorption of a photon induces a transition from level 1 to 2.

$$\frac{dN_1}{dt} = -N_1 W_{12} \qquad \frac{dN_2}{dt} = N_2 W_{21} = -\frac{dN_1}{dt}$$

  - The absorption rate W must depend on the intensity and the incident frequency. We'll represent this by the spectral energy density.
  - For light at a *specific* frequency, define

$$W_{12} = B_{12}\rho\left(\nu_0\right)$$ $B_{12}$ = Einstein "B" coefficient

  - Will generalize later for broadband light

# Spontaneous emission

- An atom in an excited state can decay to another level through radiation = spontaneous emission

$$\frac{dN_2}{dt} = -N_2 A_{21} \rightarrow N_2(t) = N_2(0) e^{-A_{21} t}$$

Lifetime of state:

$$\tau_2 = 1 / A_{21}$$

- If there are multiple destination states, rates add. Total decay out of level *i* :

$$\frac{dN_i}{dt} = -\sum_j A_{ij}$$

Lifetime of state:

$$\tau_i = 1 / \sum_j A_{ij}$$

- Note this type of process is independent of any incident light.

# Einstein's treatment of emission and absorption

- Based on thermodynamic principles, Einstein predicted the existence of stimulated emission.
- First suppose we have only absorption and spontaneous emission.
- Rate equations for a two-level system:

$$\frac{dN_1}{dt} = -N_1 B_{12}\rho(\nu) + N_2 A_{21} \qquad \frac{dN_2}{dt} = +N_1 B_{12}\rho(\nu) - N_2 A_{21}$$

- In equilibrium with the field, no net change in population densities

$$0 = -N_1^{\,e} B_{12}\rho(\nu) + N_2^{\,e} A_{21} \rightarrow \frac{N_2^{\,e}}{N_1^{\,e}} = \frac{B_{12}\rho(\nu)}{A_{21}}$$

# Thermal equilibrium with BB field

- An atom that is in thermal equilibrium has populations that follow the Boltzmann distribution:

$$\frac{N_2^e}{N_1^e} = \frac{g_2}{g_1} e^{-h\nu_{21}/k_B T} = \frac{B_{12}\rho(\nu)}{A_{21}} \rightarrow \rho(\nu) = \frac{A_{21}}{B_{12}} \frac{g_2}{g_1} e^{-h\nu_{21}/k_B T}$$

- A field in thermal equilibrium should have the blackbody spectral energy density

$$\rho_{BB}(\nu) = 8\pi \frac{\nu^2}{c^3} \frac{h\nu}{e^{h\nu/k_B T} - 1}$$

  – What we have is ok in the high frequency limit, but not fully consistent with the BB curve.

# Stimulated emission

- Things make more sense if we allow for another route for decay from 2 to 1

$$0 = -N_1^{\,e} B_{12}\rho(\nu) + N_2^{\,e} B_{21}\rho(\nu) + N_2^{\,e} A_{21} \rightarrow \frac{N_2^{\,e}}{N_1^{\,e}} = \frac{B_{12}\rho(\nu)}{A_{21} + B_{21}\rho(\nu)}$$

$$\frac{N_2^e}{N_1^e} = \frac{g_2}{g_1} e^{-h\nu_{21}/k_B T} = \frac{B_{12}\rho(\nu)}{A_{21} + B_{21}\rho(\nu)}$$

- Solve for the equilibrium spectral energy density

$$\frac{g_2}{g_1} e^{-h\nu_{21}/k_B T}\left(A_{21} + B_{21}\rho(\nu)\right) = B_{12}\rho(\nu)$$

$$\rho(\nu) = \frac{A_{21}}{B_{12}\frac{g_1}{g_2} e^{h\nu_{21}/k_B T} - B_{21}}$$

# Einstein's relations between A and B coefficients

- If both the atoms and BB cavity are in thermal equilibrium, the ρ(v)'s that satisfy that constraint must be the same

$$\rho_{BB}(\nu) = 8\pi \frac{\nu^2}{c^3} \frac{h\nu}{e^{h\nu/k_B T} - 1} \qquad \rho(\nu) = \frac{A_{21}}{B_{12}\frac{g_1}{g_2} e^{h\nu_{21}/k_B T} - B_{21}}$$

  - The two forms will have the same structure if

$$\boxed{B_{12}\frac{g_1}{g_2} = B_{21}} \rightarrow \rho(\nu) = \frac{A_{21}}{B_{21}\left(e^{h\nu_{21}/k_B T} - 1\right)}$$

  - So the processes of absorption and stimulated emission are linked.
  - Finally, for $\rho_{BB}(\nu) = \rho(\nu)$

$$\boxed{A_{21} = \frac{8\pi h\nu^3}{c^3} B_{21}}$$

# Physical significance of A/B

- Dimensionally, $B_{21}\rho$ gives a rate, so in the relation between A and B, $A_{21} = \frac{8\pi h \nu^3}{c^3} B_{21}$

$\rho(\nu) = \frac{8\pi h \nu^3}{c^3}$ is a type of spectral energy density.

In QED, the E and B energy densities are quantized, and the quanta are the photons.

$\rho(\nu) = \frac{8\pi h \nu^3}{c^3}$ is effectively the spectral energy density of the vacuum fluctuations of the field.

# Connect intensity changes to atomic rates

- In a volume V, absorbed power is $\frac{dP_a}{dV} = W_{12} N_1 h\nu$

- For a beam with area A, $\frac{dP_a}{dV} = \frac{1}{A}\frac{dP}{dz} = -\frac{dI}{dz}$

- Intensity and energy density are related: $\rho c = I$

$$\frac{dP_a}{dV} = -\frac{dI}{dz} = B_{12} \rho N_1 h\nu \qquad \frac{dI}{dz} = -I\, N_1 \frac{B_{12} h\nu}{c} = -I\, N_1 \sigma_{12}$$

$$\boxed{\sigma_{12} = \frac{B_{12} h\nu}{c}}$$

Will generalize this to account for lineshape of absorption, and bandwidth of source.

Note that the mean free path of photons in the medium is 1/α

# Optical gain

- With population in both levels 1 and 2,

$$\frac{dI}{dz} = I\left(N_2 B_{21} - N_1 B_{12}\right)\frac{h\nu_{21}}{c}$$

$$B_{12}\frac{g_1}{g_2} = B_{21}$$

$$\frac{dI}{dz} = I\left(N_2 - N_1\frac{g_2}{g_1}\right)\frac{B_{21}h\nu_{21}}{c} = I\,N_{inv}\sigma_{21}$$

Inversion density

Gain cross-section

$$\boxed{I(z) = I_0 e^{gz}}$$

g: gain coefficient = $N_{inv}$ $\sigma_{21}$
(opposite sign from absorption coefficient)

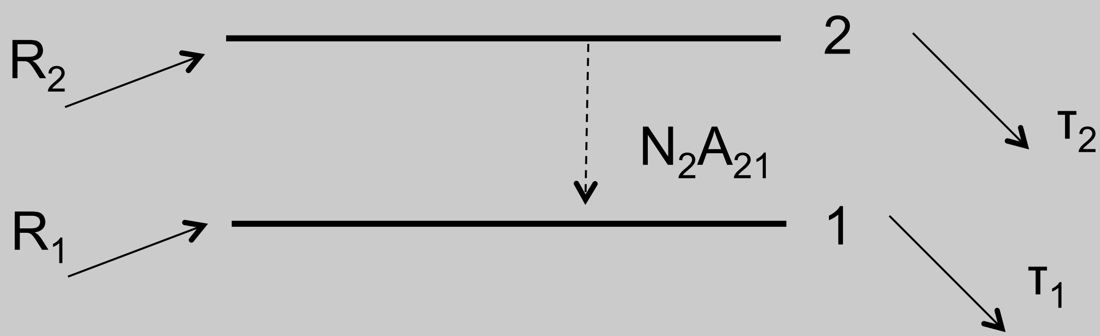
For an amplifier of length L,

$$\boxed{I(L) = I_0 e^{gL} = I_0 G_0}$$

$G_0$: small signal single-pass gain

# General conditions for steady-state inversion (gain)

- Consider general situation, including pumping rates $R_1$, $R_2$ and lifetimes $\tau_1$, $\tau_2$




  - Lifetime of level 2 includes $1/A_{21}$, but also includes decay to other levels

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2}$$

$$\frac{dN_1}{dt} = R_1 + N_2 A_{21} - \frac{N_1}{\tau_1}$$

Both rates go to zero in steady state

# Steady-state inversion

- Solve for inversion density

$$0 = R_2 - \frac{N_2}{\tau_2} \rightarrow N_2 = R_2\tau_2$$

$$0 = R_1 + N_2 A_{21} - \frac{N_1}{\tau_1} \rightarrow N_1 = R_1\tau_1 + N_2 A_{21}\tau_1$$

$$N_1 = \tau_1\left(R_1 + R_2\tau_2 A_{21}\right)$$

- For gain, $\frac{N_2}{g_2} > \frac{N_1}{g_1}$

$$\frac{R_2\tau_2}{g_2} > \frac{\tau_1}{g_1}\left(R_1 + R_2\tau_2 A_{21}\right) \rightarrow \frac{R_2\tau_2}{g_2} - \frac{R_2\tau_2 A_{21}\tau_1}{g_1} > \frac{R_1\tau_1}{g_1}$$

$$R_2\tau_2\left(\frac{1}{g_2} - \frac{A_{21}\tau_1}{g_1}\right) > \frac{R_1\tau_1}{g_1}$$

$$\boxed{\frac{R_2\tau_2}{R_1\tau_1}\frac{g_1}{g_2}\left(1 - \frac{g_2}{g_1}A_{21}\tau_1\right) > 1}$$

# Interpretation of conditions for steady-state gain

$$\frac{R_2\tau_2}{R_1\tau_1}\frac{g_1}{g_2}\left(1-\frac{g_2}{g_1}A_{21}\tau_1\right)>1$$

- Selective pumping: $R_2 > R_1$
- Favorable lifetime ratio: $\tau_2 > \tau_1$
- Favorable degeneracy ratio: $g_1 > g_2$
- Necessary condition:

$$A_{21} < \frac{g_1}{g_2}\frac{1}{\tau_1}$$

  - Lower level has to empty out faster than spontaneous emission fills it.
  - *No CW 3-level system, but transient pumping is ok*