Introduction to Fourier optics

Definitions: our signals are the complex fields

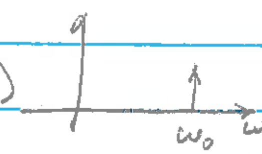
t-ω space $f(t)$, $F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \equiv \mathcal{F}\{f(t)\}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega = \mathcal{F}^{-1}\{F(\omega)\}$$

conventions:

1) ω not ν as frequency variable
2) $e^{+i\omega t}$ on forward transform
   - so that $\mathcal{F}\{e^{-i\omega_0 t}\} = \delta(\omega-\omega_0)$



3) $\frac{1}{2\pi}$ in front of $\mathcal{F}^{-1}\{\ \}$

x-f_x space

$$F(f_x,f_y) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-2\pi(xf_x+yf_y)}\, dx\,dy = \mathcal{F}\{f(x,y)\}$$

$$f(x,y) = \iint_{-\infty}^{\infty} F(f_x,f_y)\, e^{+2\pi(xf_x+yf_y)}\, df_x\,df_y$$

1) f_x not k_x as spatial frequency (Goodman conv.)
2) opposite sign convention from t,ω
   - so that $\mathcal{F}\{e^{+ik_x x}\} \to \delta(f_x - k_x/2\pi)$
3) no $1/2\pi$ (comes from using f_x variable)

Why Fourier representation?
physics of linear propagation is usually separable in one domain.

example: dispersive propagation $E_{out}(\omega) = E_{in}(\omega) e^{i\phi(\omega)}$

$$\phi(\omega) = \frac{\omega}{c} n(\omega) L$$

diffractive propagation:

$$E_{out}(x,y,z) = \mathcal{F}^{-1}\left\{ E_{in}(f_x,f_y) e^{i\phi(f_x,f_y)} \right\}$$

$$\phi(f_x,f_y) = \frac{2\pi}{\lambda}\sqrt{1-\lambda^2 f_x^2 - \lambda^2 f_y^2}\, z$$

- linear propagation takes place in ω, f_x, f_y space.
- we often measure (or do experiments) in t, x, y space.
  - measure spectrum with spectrometer $I(\omega) \propto |E(\omega)|^2$
  - most measurements are on time-averaged intensity

Research examples:
- pulse compression
- pulse shaping
- spatial filtering
- image processing
- lens design
- light scattering, acousto-optics
- spatio-temporal coupling
- resonators
- guided waves
- Airy waves

Related domains:
- cylindrical coords → Fourier-Bessel, Hankel xform
  - > Bessel beams
- Wigner xform: mix t-ω space

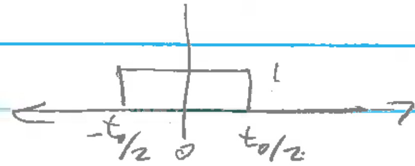
Techniques

- analytic / manual: application of FT pairs, theorems
  → graphical, intuitive understanding of physics + math

- analytic / Mathematica
  > does not easily simplify in terms of xform pairs
  > useful when functions are cumbersome to do by hand.

- numeric / FFT, convolution
  > many real functions cannot be xformed analytic
  > data analysis
  > modeling.

$\mathrm{rect}(t/t_0) \equiv$



$\mathcal{F}\{\mathrm{rect}(t/t_0)\} = ?$

impt: implement rect by changing limits on integral

$$= \int_{-t_0/2}^{t_0/2} e^{i\omega t}\,dt = \frac{1}{i\omega} e^{i\omega t_0/2} - \frac{1}{i\omega} e^{-i\omega t_0/2}$$

convert to sin( )

$$= \frac{2}{\omega}\frac{1}{2i}\left(e^{i\omega t_0/2} - e^{-i\omega t_0/2}\right)$$

$$= \frac{2}{\omega}\sin(\omega t_0/2) = t_0 \frac{\sin\left(\frac{\omega t_0}{2}\right)}{\frac{\omega t_0}{2}}$$

$$\equiv t_0\,\mathrm{sinc}\left(\frac{\omega t_0}{2}\right)$$

Fourier transform pairs

Gaussian

use $\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$ even if z is complex.

$$f(t) = e^{-t^2/t_0^2}$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2}\, e^{i\omega t}\, dt$$

complete square in exponent:

$$\frac{t^2}{t_0^2} - i\omega t = \frac{1}{t_0^2}\left(t^2 - i\omega t t_0^2\right) = \underbrace{\frac{1}{t_0^2}\left(t - i\frac{\omega t_0^2}{2}\right)^2}_{=z^2} + \frac{\omega^2 t_0^2}{4}$$

$$dz = \frac{1}{t_0}\, dt \rightarrow F(\omega) = e^{-\frac{\omega^2 t_0^2}{4}}\, t_0 \int_{-\infty}^{\infty} e^{-z^2}\, dz$$

$$= \sqrt{\pi}\, t_0 \exp\left[-\omega^2 t_0^2/4\right]$$

time-bandwidth product

$$t_0 \cdot \delta\omega = t_0 \cdot \frac{2}{t_0} = 2$$

$t_0, \delta\omega$ = $1/e$ halfwidths in field.

FWHM (more common)

$$I(t) = |f(t)|^2 = e^{-a t^2/\tau^2}$$

at $t = \tau/2$ $I = \frac{1}{2} = e^{-a/4} \rightarrow \ln 2 = a/4$

$$a = 4 \ln 2$$

$$f(t) = e^{-2\ln 2\, t^2/\tau^2}$$

converting back to field.

$\therefore$ $t_0 = \tau/\sqrt{2\ln 2}$ similarly $\delta\omega = \Delta\omega/\sqrt{2\ln 2}$

now $\tau\Delta\omega = 2\ln 2\, t_0\, \delta\omega = 4\ln 2 = 2.77$

$$\tau\Delta\nu = \frac{4\ln 2}{2\pi} = 0.44$$