# Heat Equ on a bounded Region of $\mathbb{R}^{2+1}$.

(1) $$\frac{\partial u}{\partial t} = c^2 \Delta u = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \quad \begin{matrix} t\in(0,\infty) \\ x\in(0,L_x) \\ y\in(0,L_y)\end{matrix}$$

(2) $u(0,y,t)=0, \ u(L_x,y,t)=0$

$u_y(x,0,t)=0, \ u_y(x,L_y,t)=0$

(3) $u(x,y,0)=f(x,y)$

Step 1: Separation of Variables

$$u(x,y,t) = G(t)\,F(x,y) \Rightarrow \frac{\partial u}{\partial t} = G'F, \quad \frac{\partial^2 u}{\partial x^2} = GF_{xx}$$

$$\frac{\partial^2 u}{\partial y^2} = GF_{yy}$$

$$\Rightarrow (1) \Leftrightarrow \frac{G'}{c^2 G} = \frac{F_{xx}+F_{yy}}{F} = -\lambda \in \mathbb{R}$$

$$\Rightarrow \quad G' = -\lambda c^2 G$$

$$F_{xx}+F_{yy}+\lambda F = 0$$

Assume now that

$$F(x,y) = X(x)\,Y(y) \Rightarrow F_{xx} = X''Y, \ F_{yy} = XY''$$

$$\Rightarrow X''Y + Y''X + \lambda XY = 0 \Leftrightarrow \frac{X''}{X} = -\left(\frac{Y''}{Y}+\lambda\right) = -k \in \mathbb{R}$$

$$\Rightarrow \quad \left.\begin{array}{l} G' = -\lambda c^2 G \\ X'' + kX = 0 \\ Y'' + pY = 0, \quad p = \lambda - k \end{array}\right\} (*)$$

Step 2: Convert (2) + Solve BVPs + ODE.

Note:

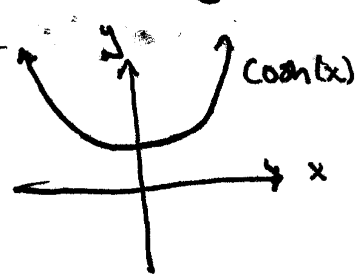
- $u(0,y,t) = G(t)X(0)Y(y) = 0 \Rightarrow X(0) = 0$, for a nontrivial state.

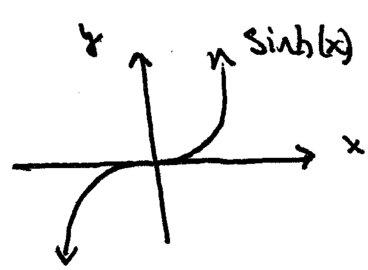
$$\Rightarrow (2) \Rightarrow \quad X'(0) = 0, \quad X(L_x) = 0$$
$$Y'(0) = 0, \quad Y'(L_y) = 0$$

- $(*)$ is of the form
$$y'' + \lambda y = 0, \quad \lambda \in \mathbb{R}$$
which has sol$\underline{n}$

$$\lambda > 0: \quad y_1(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$
$$\lambda < 0: \quad y_2(x) = c_3 \cosh(\sqrt{|\lambda|}\, x) + c_4 \sinh(\sqrt{|\lambda|}\, x)$$
$$\lambda = 0: \quad y_3(x) = c_5 + c_6 x$$

where



With this in mind we consider the two BVP

$$\begin{cases} X'' + kX = 0 \\ X(0) = 0, \ X(L_x) = 0 \end{cases} \Rightarrow X_n(x) = \sin(\sqrt{k_n}\, x), \quad \sqrt{k_n} = \frac{n\pi}{L_x}, \ n = 1, 2, 3, \ldots$$

$$\begin{cases} Y'' + pY = 0 \\ Y'(0) = 0, \ Y'(L_y) = 0 \end{cases} \Rightarrow Y_m(y) = \cos(\sqrt{p_m}\, y), \quad \sqrt{p_m} = \frac{m\pi}{L_y}, \ m = 0, 1, 2, \ldots$$

$$\Rightarrow \quad \lambda_{nm} = p_m + k_n = \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_x}\right)^2,$$

$$\Rightarrow \quad G_{nm}' = -c^2 \lambda_{nm} G_{nm} \Rightarrow G_{nm}(t) = B_{nm} e^{-c^2 \lambda_{nm} t}$$

<u>Step 3:</u> Apply superposition and initial conditions

$$u(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} u_{nm}(x,y,t) =$$

$$= \sum_{n=1}^{\infty} B_{n0} \sin(\sqrt{k_n}\, x) e^{-c^2 \lambda_{n0} t} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{nm} e^{-c^2 \lambda_{nm} t} \sin(\sqrt{k_n}\, x) \cos(\sqrt{p_m}\, y)$$

(*) General Sol<u>n</u>.

$\cancel{A_0(x) + \sum_{m=1}^{\infty} A_m}$

The initial condition gives

$$f(x,y) = A_0(x) + \sum_{m=1}^{\infty} A_m(x)\cos(\sqrt{p_m}\, y) \qquad (\ddot\smile)$$

where

$$A_0(x) = \sum_{n=1}^{\infty} B_{n0}\sin(\sqrt{k_n}\, x) \Rightarrow B_{n0} = \frac{2}{L_x}\int_0^{L_x} A_0(x)\sin(\sqrt{k_n}\, x)\,dx$$

and

$$A_m(x) = \sum_{n=1}^{\infty} B_{nm}\sin(\sqrt{k_n}\, x) \Rightarrow B_{nm} = \frac{2}{L_x}\int_0^{L_x} A_m(x)\sin(\sqrt{k_n}\, x)\,dx$$

but from $(\ddot\smile)$ we also have

$$A_0(x) = \frac{1}{L_y}\int_0^{L_y} f(x,y)\,dy$$

and

$$A_n(x) = \frac{2}{L_y}\int_0^{L_y} f(x,y)\cos(\sqrt{p_m}\, y)\,dy$$

Thus

$$B_{n0} = \frac{2}{L_x L_y}\int_0^{L_x}\int_0^{L_y} f(x,y)\sin(\sqrt{k_n}\, x)\,dy\,dx$$

$$B_{nm} = \frac{4}{L_x L_y}\int_0^{L_x}\int_0^{L_y} f(x,y)\cos(\sqrt{p_m}\, y)\sin(\sqrt{k_n}\, x)\,dy\,dx$$