Note Title 4/14/2008

For test on Friday

1) one more (3 total cheat sheets)

2) I will give you any integral that you need.

3) extra time

4) The relevant parts of tables 4.3, 4.4 4.6

Wednesday we will Review ch. 4.

4_14_08

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4.8 b)

Compute the first few energy levels for the infinite spherical potential with $\ell = 1$.

General solution of radial equation is $r R(r) \equiv u(r) = A r j_\ell(kr) + B r N_\ell(kr)$

The N_ℓ terms blow up at the origin, so

$$R(r) = A j_\ell(kr)$$

I will give you a table of j_ℓ:

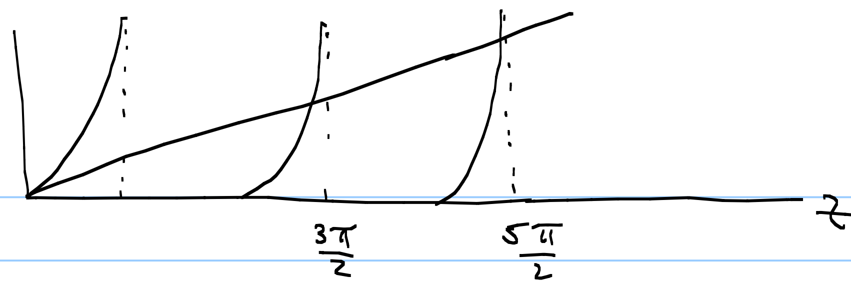
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

So in order that $R(a) = 0$ must have

$$j_1(ka) = \frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka} = 0$$

$$\Rightarrow \quad \tan ka = ka$$

Plot $\tan z$ vs z



So $\quad ka \cong \frac{\pi}{2}(2N+1) \qquad N = 0, 1, 2, \ldots$

$\quad\quad\;\; = \pi\left(N+\frac{1}{2}\right) \qquad N = 0, 1, 2, \ldots$

Now $\quad k = \sqrt{2mE}\,/\hbar$

so $\quad E_N = \dfrac{k_N^2\hbar^2}{2m} = \dfrac{\hbar^2\pi^2}{2ma^2}\left(N+\frac{1}{2}\right)^2$

4.13 $\quad$ find $\langle r \rangle$ and $\langle r^2 \rangle$ for the ground state of Hydrogen.

$$\psi_{n\ell m} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}}\; e^{-r/na} \left(\frac{2r}{na}\right)^{\ell} L^{2\ell+1}_{n-\ell-1}(2r/na)$$

$$\times\, Y_\ell^m(\theta, \varphi)$$

Ground state $\quad n=1 \quad \ell=0 \quad m=0$

$$\sqrt{\left(\frac{2}{a}\right)^3 \frac{1}{2}}\; e^{-r/a}\; \underbrace{L^1_0(2r/a)}_{1}\; \underbrace{Y_0^0}_{\frac{1}{\sqrt{4\pi}}}$$

table 4.6

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}$$

$$\langle r^N \rangle = \frac{1}{\pi a^3} \int r^N e^{-2r/a} \, dV$$

$$= \frac{4\pi}{\pi a^3} \int_0^\infty r^{N+2} e^{-2r/a} \, dr$$

$$\frac{2r}{a} = \rho \qquad dr = \frac{a}{2} d\rho$$

$$r = \frac{a}{2} \rho$$

$$\langle r^N \rangle = \frac{4}{a^3} \frac{a}{2} \left(\frac{a}{2}\right)^{N+2} \int_0^\infty \rho^{N+2} e^{-\rho} \, dr$$

I would give you this

$$\langle r^1 \rangle = \frac{4}{a^3} \frac{a}{2} \left(\frac{a}{2}\right)^3 \boxed{\underbrace{\int_0^\infty \rho^3 e^{-\rho} \, d\rho}_{= 3!}} = \frac{a}{4} 3!$$

$$= \frac{3}{2} a$$

$$\langle r^2 \rangle = \frac{4}{a^3} \frac{a}{2} \left(\frac{a}{2}\right)^4 4! = a^2 \frac{4!}{8} = 3a^2$$

$$\langle x \rangle = 0, \qquad \langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle = a^2$$

4.14 what is the most probable value of r for the G.S. of Hyd.

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

So the probability of finding r between r and r+dr is

$$\underset{\sim}{4\pi} |\psi_{100}|^2 r^2 dr \equiv P(r)\, dr$$

comes from integrating out angles

$$\text{So } P(r) = 4\pi \frac{1}{\pi a^3} r^2 e^{-2r/a}.$$

$$\frac{dp}{dr} = 0 \iff 2r e^{-r/a} - r^2 \frac{2}{a} e^{-2r/a} = 0$$

$$\Rightarrow 2r = 2r^2/a \Rightarrow \boxed{r = a}$$

4.15 $\Psi(\vec{r},0) = \frac{1}{\sqrt{2}}(\psi_{211} + \psi_{21-1})$

$$\Psi(\vec{r},t) = \frac{1}{\sqrt{2}}\left(\psi_{211} e^{-iE_2 t/\hbar} + \psi_{21-1} e^{-iE_2 t/\hbar}\right)$$

$$= \frac{1}{\sqrt{2}}(\psi_{211} + \psi_{21-1}) e^{-iE_2 t/\hbar}$$

$$E_2 = \frac{E_1}{4} = -\frac{\hbar^2}{8ma^2}$$

$$\psi_{211} + \psi_{21-1} = \frac{-1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta$$
$$(e^{i\varphi} - e^{-i\varphi})$$

$$= -\frac{1}{\sqrt{2\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta \left(2i \sin\varphi\right)$$

$$\boxed{\Psi(\vec{r},t) = -\frac{1}{\sqrt{2\pi a}} \frac{i}{4a^2} r e^{-r/2a} \sin\theta \sin\varphi\, e^{-iE_2 t/\hbar}}$$

find $\langle V \rangle$

$$\langle V \rangle = \int |\psi|^2 \left(\frac{-e^2}{4\pi\epsilon_0 r}\right) dV$$

$$= \frac{1}{2\pi a} \frac{1}{16a^4} \frac{-e^2}{4\pi\epsilon_0} \int r^2 e^{-r/a} \sin^2\theta \sin^2\varphi \frac{1}{r} r^2 \sin\theta \, d\theta \, d\varphi \, dr$$

$$\frac{1}{32\pi a^5} \frac{-\hbar^2}{ma} \underbrace{\underbrace{\int_0^\infty r^3 e^{-r/a} dr}_{3!\,a^4} \underbrace{\int_0^\pi \sin^3\theta \, d\theta}_{\frac{4}{3}} \underbrace{\int_0^{2\pi} \sin^2\varphi \, d\varphi}_{\pi}}_{8\pi a^4}$$

$$\frac{e^2}{4\pi\epsilon_0} = \frac{\hbar^2}{ma}$$

$$= \frac{-\hbar^2 \, 8\pi a^4}{32\pi m a^6} = \frac{-\hbar^2}{4ma^2} = -\frac{1}{2} E_1$$