Stress Tensor for E M Field

generalize Lorentz force law

$$\vec{F} = \int_V \left(\rho \vec{E} + \frac{1}{c}\vec{J} \times \vec{B}\right) dv$$

use Max. Eqns, $\rho \to \frac{1}{4\pi} \nabla \cdot \vec{E}$

$$\frac{1}{c}\vec{J} \to \frac{1}{4\pi}\left[\nabla \times B - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right]$$

We want to represent F in terms of fields only.

note combination $\vec{V}_1 \vec{V}_2$ is a tensor

"outer product"

will show $\to$ stress-energy tensor

$$T_{ij} = \frac{1}{4\pi}\left[\left(E_i E_j + B_i B_j\right) - \frac{1}{2}\left(E^2 + B^2\right)\delta_{ij}\right]$$

Terms are dimensionally ~ energy density

$$\vec{F} = \int_V \underbrace{\frac{1}{4\pi}\left[(\nabla \cdot \vec{E})\vec{E} + \left(\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right) \times \vec{B}\right]}_{\text{force/volume} \equiv \vec{f}} dv$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\vec{E} \times \vec{B}) = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \times \vec{B} + \frac{1}{c}\vec{E} \times \frac{\partial \vec{B}}{\partial t}$$

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\to \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \times B = \frac{1}{c}\frac{\partial}{\partial t}(\vec{E} \times \vec{B}) + \vec{E} \times (\nabla \times \vec{E})$$

Now force/volume is

$$\vec{f} = \frac{1}{4\pi}\left[(\nabla \cdot \vec{E})\vec{E} - \vec{E} \times (\vec{\nabla} \times \vec{E}) - \vec{B} \times (\nabla \times \vec{B}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E} \times \vec{B}) + \underbrace{(\nabla \cdot \vec{B})}_{=0}\vec{B}\right]$$

↳ permutation sign change

continue rearrangement:

$$\nabla(E^2) = 2(\vec{E}\cdot\vec{\nabla})\vec{E} + 2\underbrace{\vec{E}\times(\nabla\times\vec{E})}_{\text{solve for this}}$$

$$\vec{E}\times\nabla\times\vec{E} = \frac{1}{2}\nabla(E^2) - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

same for $\vec{B}$

$$\vec{f} = \frac{1}{4\pi}\Big[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} + (\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B}$$

$$- \frac{1}{2}\nabla(E^2 + B^2) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\Big]$$

this is a mess, but we will write this as

$$\vec{f} = \nabla\cdot\overleftrightarrow{T} - \frac{1}{c^2}\frac{\partial\vec{S}}{\partial t} \qquad \vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$$

$\frac{1}{c^2}\vec{S}$ = momentum density

Maxwell Stress tensor

$$T_{ij} \equiv \frac{1}{4\pi}\Big(E_iE_j - \frac{1}{2}\delta_{ij}\underbrace{E^2}_{E_x^2+E_y^2+E_z^2} + B_iB_j - \frac{1}{2}\delta_{ij}B^2\Big)$$

$$T_{xx} = \frac{1}{4\pi}\Big[\frac{1}{2}(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2}(B_x^2 - B_y^2 - B_z^2)\Big]$$

$$T_{xy} = \frac{1}{4\pi}(E_xE_y + B_xB_y)$$

$$\vec{a}\cdot\overleftrightarrow{T} = \vec{b} \qquad (\vec{a}\cdot\overleftrightarrow{T})_j = \sum_i a_iT_{ij}$$

$$(a_1\ a_2\ a_3)\begin{pmatrix} 11 & 12 & 13 \\ 21 & & \text{etc.} \\ 31 & & \end{pmatrix} = \begin{matrix} b_1 \\ b_2 \\ \end{matrix}$$

Along these lines,

$$(\nabla\cdot\overleftrightarrow{T})_j = \frac{1}{4\pi}\sum_i \frac{\partial}{\partial x_i} T_{ij} = \sum_i \frac{\partial}{\partial x_i}\left(E_i E_j - \frac{1}{2}E^2\right) + B\text{-terms}$$

$$= \sum_i \left(\frac{\partial E_i}{\partial x_i} E_j + E_i \frac{\partial E_j}{\partial x_i}\right) - \frac{1}{2}\frac{\partial}{\partial x_i}(E^2) + B's$$

$$= (\nabla\cdot\vec{E})E_j + (\vec{E}\cdot\vec{\nabla})E_j - \frac{1}{2}\nabla_j E^2 + B's$$

which is what we have.

Return to the total force:

$$\vec{F} = \int_V \left(\nabla\cdot\overleftrightarrow{T} - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}\right) dv$$

$$= \oint_S \overleftrightarrow{T}\cdot d\vec{a} - \frac{1}{c^2}\frac{d}{dt}\int \vec{S}\, dv$$

$\overleftrightarrow{T}$ = Force/area = pressure or stress

this gives force on what is enclosed by surface.

this force is acting on charges + currents

$$\rightarrow \vec{F} = \frac{d}{dt}\vec{P}_{mech}$$

identify momentum of field

$$\vec{P}_{EM} = \frac{1}{c^2}\int_V \vec{S}\, dv$$

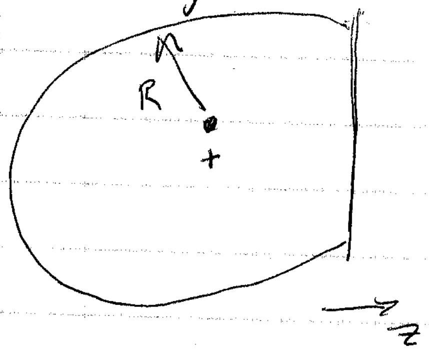
densities

$\rightarrow$ conservation of momentum: $-\nabla\cdot\overleftrightarrow{T} + \frac{\partial}{\partial t}\left(\mathcal{P}_{mech} + \mathcal{P}_{EM}\right) = 0$

Stress tensor example: force between two charges

force/volume $\vec{f} = \nabla \cdot \overleftrightarrow{T} - \cancel{\frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}}$

$\vec{F} = \int \vec{f}\, dv = \oint \overleftrightarrow{T} \cdot d\vec{a}$ $\quad 0$ (static)



how do we choose the surface?

Note force is also given by

$F = \int \left(\rho \vec{E} + \frac{\vec{J}}{c} \times \vec{B}\right) d^3x$

the volume must include the charges + currents affected,

$\therefore$ choose easiest

here, we can let $R \to \infty$, and contributions on hemisphere $\to 0$

left with integral on plane of symmetry at $z=0$

$$d\vec{F} = \begin{pmatrix} dF_x \\ dF_y \\ dF_z \end{pmatrix} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} da_x \\ da_y \\ da_z \end{pmatrix}$$

Anticipate $\vec{F} = F\hat{z}$ so calc $(\overleftrightarrow{T} \cdot d\vec{a})_z$

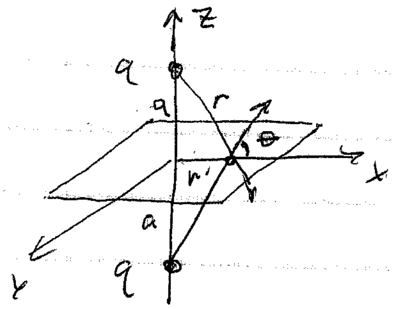
But $da_x = da_y = 0$ on $z=0$ surface, so

$$\vec{F} = \int_{z=0 \text{ plane}} T_{zz}\, dx\, dy \; \hat{z}$$

Calculate $\overleftrightarrow{T}$: $\quad$ B's all $= 0$

$$\vec{E} = \frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2$$ use <u>total</u> e-field

on plane, $\vec{E} = \frac{2q}{r^2}\cos\theta\, \hat{r}'$ $\quad q_1 = q_2 \quad \cos\theta = \frac{r'}{r}$



$$T_{zz} = \frac{1}{4\pi}\left(E_z^2 - \frac{1}{2}E^2\right)$$

here $E_z = 0$ (on the plane)

$$T_{zz} = \frac{1}{4\pi}\left(-\frac{1}{2}\left(2q\,\frac{r'}{r^3}\right)^2\right)$$

$$= \frac{-q^2 r'^2}{2\pi(r'^2+a^2)^3}$$

$$da_z = -r'dr'd\phi \quad \text{(upper charge)}$$

$$\therefore\ F_z = +\frac{1}{2\pi}q^2 \int_0^{2\pi} d\phi \int_0^{\infty} dr' \frac{r'^3}{(r'^2+a^2)^3} \rightarrow \frac{q^2}{(2a)^2} \quad \checkmark$$

See also: math notebook stress tensor - pt charges.nb