

Day 8: Cylindrical Symmetry & Wrap Up / Review

Much of 4.4 is kind of junk (handwavy "proofs", esoteric & poorly explained examples).

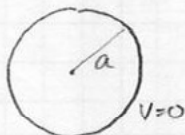
The real take-home point is that much of what you know about solving problems with spheres transfers readily to cylinders. For example:

↑ ↑ ↑ ↑ \vec{E}

Grounded cylinder in external field. Z axis is axis of cylinder
No z dependence

Let's let \vec{E} lie in the $\phi=0$ direction

For large r , $V = -E_0 r \cos\phi$



Now we solve $\nabla^2 V = 0$ in cylindrical, sans z.
We take some basic building blocks, including cylindrical harmonics, & work with

↑ ↑ ↑ ↑

$$V(r, \phi) = A \ln(r) + B + \frac{C \cos\phi}{r} + D r \cos\phi$$

B is arbitrary, so $B=0$. A is probably zero (we likely need ϕ -dependence).

$$D = -E_0$$

Can we make $V(a, \phi) = 0$ with

$$-E_0 a \cos\phi + \frac{C \cos\phi}{a} ?$$

$$\frac{C}{a} = E_0 a \Rightarrow C = E_0 a^2$$

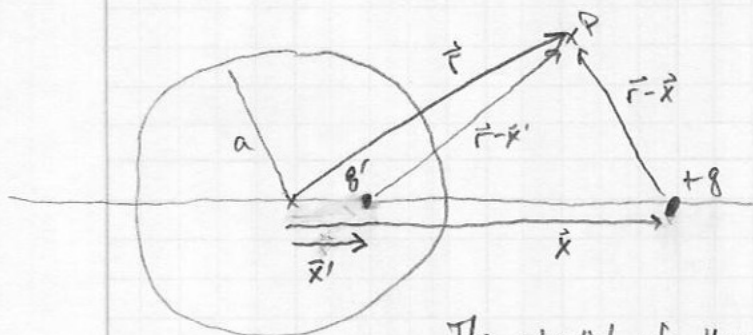
And thus
$$V(r, \phi) = -E_0 r \cos\phi + E_0 a^2 \frac{\cos\phi}{r}$$

From there \vec{E} and σ follow.

Line by line, nearly identical to the grounded sphere.
But this is nothing new; spherical & cylindrical problems often differ only slightly (see: Phys 200 Gauss's Law problems).

Let's do another classic image problem in spherical coords. A grounded conducting sphere with a point charge outside. Find the voltage outside.

Given the symmetry we may as well put the charge on the x-axis. Now, if by some miracle we can do this with an image charge, the charge needs to be



- A) In the sphere (can't fit with ρ in $-\nabla^2 V = \rho/\epsilon_0$)
 B) On the x-axis

The potential of these two point charges would be

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{|\vec{r}-\vec{x}|} + \frac{q'}{|\vec{r}-\vec{x}'|} \right]$$

This satisfies $-\nabla^2 V = \rho/\epsilon_0$ for $r \geq a$, so if we can choose \vec{x}' and q' to make $V(a) = 0$, we win.

Drop constants & put \vec{r} on the sphere:

$$\left[\frac{q}{|a\hat{r}-\vec{x}|} + \frac{q'}{|a\hat{r}-\vec{x}'|} \right]$$

Pull some stuff out to try to break this into two eqns:

$$\left[\frac{q}{a} \cdot \frac{1}{|\hat{r}-\frac{\vec{x}}{a}|} + \frac{q'}{x'} \cdot \frac{1}{|\frac{a}{x'}\hat{r}-\hat{i}|} \right] = 0?$$

We'd need $\frac{q}{a} = -\frac{q'}{x'}$ and $|\hat{r}-\frac{\vec{x}}{a}| = |\frac{a}{x'}\hat{r}-\hat{i}|$ or

$$|\hat{r}-\frac{\vec{x}}{a}|^2 = |\frac{a}{x'}\hat{r}-\hat{i}|^2$$

$$(\hat{r}-\frac{\vec{x}}{a}) \cdot (\hat{r}-\frac{\vec{x}}{a}) = 1 - \frac{2x}{a}(\hat{r} \cdot \hat{i}) + \frac{x^2}{a^2} = \frac{a^2}{x^2} - \frac{2a}{x}(\hat{r} \cdot \hat{i}) + 1 = (\frac{a}{x'}\hat{r}-\hat{i}) \cdot (\frac{a}{x'}\hat{r}-\hat{i})$$

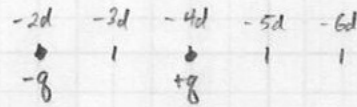
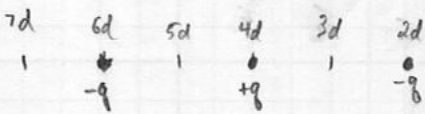
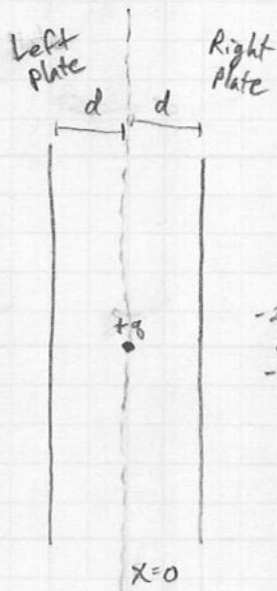
$$\Rightarrow \frac{x^2}{a^2} = \frac{a^2}{x'^2} \quad \text{or} \quad \frac{x}{a} = \frac{a}{x'}$$

So $xx' = a^2$ (x, x' called conjugate points) $x' = \frac{a^2}{x}$

$$q' = \frac{qx'}{a} = -\frac{qa}{x}$$

Run those back into $V(\vec{r})$ and we've got it. Can then get \vec{E} & σ in the usual way.

Close out with another planar problem. Two grounded parallel planes separated by $2d$, with a charge $+q$ in the middle of them. Let's see if we can build an image solution by satisfying the BCs, $V=0$ at each plate.



Attempt 1: Place $-q$ at $2d$.
This makes $V_{\text{left}}=0$,
 $V_{\text{right}} = +1 - 1/3 = 2/3$ units

Attempt 2: Place $-q$ at $-2d$ and
 $+q$ at $-4d$. $V_{\text{right}}=0$
 $V_{\text{left}} = 1 - 1/3 + 1/5 = -2/15$ unit
Closer!

Attempt 3: Place $+q$ at $4d$, $-q$ at $6d$
 $V_{\text{left}}=0$
 $V_{\text{right}} = +1/5 - 1/7 = 2/35$ units

If we keep doing this, V quickly converges to zero at either plane. We have infinitely many images, and the voltage in the space between the planes at any point on the $x=0$ plane comes from:

$$V(r) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r} + \sum_{k=1}^{\infty} \frac{2(-1)^k}{\sqrt{r^2 + (2dk)^2}} \right]$$

Term for real charge,
 $r = \sqrt{y^2 + z^2}$

images come in pairs
alternate in signs
 x^2 , each pair is $2d$ farther out than the previous.

And from here, we could get σ and \vec{E} .