Advanced Engineering Mathematics	**Homework One**
Systems of Linear Equations : Algebra, Geometry, Row-Reduction, Determinants, Transformations	
Text: 7.1-7.3, 7.5, 7.7-7.8	Lecture Slides: 2-4

Quote of Homework One	
Paul Atreides: Fear is the mind-killer. Fear is the little-death that brings total obliteration.	
	Frank Herbert : Dune (1965)

1. Matrix Multiplication

Define the *commutator* and *anti-commutator* of two square matrices to be,

$$[\cdot,\cdot] : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, \text{ such that } [\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA}, \quad \text{for all } \mathbf{A},\mathbf{B} \in \mathbb{C}^{n\times n},$$
$$\{\cdot,\cdot\} : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, \text{ such that } \{\mathbf{A},\mathbf{B}\} = \mathbf{AB} + \mathbf{BA}, \quad \text{for all } \mathbf{A},\mathbf{B} \in \mathbb{C}^{n\times n},$$

respectively. Also define the *Kronecker delta* and *Levi-Civita* symbols to be,

$$\delta_{ij} : \mathbb{N}\times\mathbb{N} \to \{0,1\}, \text{ such that } \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

$$\epsilon_{ijk} : (i,j,k) \to \{-1,0,1\}, \text{ such that } \epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1) \text{ or } (3,1,2), \\ -1, & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2) \text{ or } (2,1,3), \\ 0, & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

respectively. Also define the so-called *Pauli spin-matrices* (PSM) to be,

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

1.1. **The PSM are *self-adjoint* matrices.** Show that $\sigma_m = \sigma_m^{\mathrm{H}}$ for $m = 1,2,3$.

1.2. **The PSM are *unitary* matrices.** Show that $\sigma_m^2 = \mathbf{I}$ for $m = 1,2,3$ where $[\mathbf{I}]_{ij} = \delta_{ij}$.

1.3. **Trace and Determinant.** Show that $\mathrm{tr}(\sigma_m) = 0$ and $\det(\sigma_m) = -1$ for $m = 1,2,3$.

1.4. **Anti-Commutation Relations.** Show that $\{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbf{I}$ for $i = 1,2,3$ and $j = 1,2,3$.

1.5. **Commutation Relations.** Show that $[\sigma_i,\sigma_j] = 2\sqrt{-1}\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k$ for $i = 1,2,3$ and $j = 1,2,3$.

2. Solutions Sets to Linear Systems of Algebraic Equations

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 1 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 6 & 18 & -4 \\ -1 & 3 & 8 \\ 5 & 15 & -9 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 20 \\ 4 \\ 11 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \quad \mathbf{b}_5 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}.$$

2.1. **Algebra.** Find all solutions to $\mathbf{A}_i\mathbf{x} = \mathbf{b}_i$ for $i = 1,2,3,4,5$.

2.2. **Geometry.** Describe or plot the geometry formed by the linear systems and their solution sets.

3. Square Coefficient Data and Matrix Inversion

Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

3.1. **Matrix Inverse: Take One.** Find $\mathbf{A}^{-1}$ using the Gauss-Jordan Method. (pg.317)

3.2. **Matrix Inverse: Take Two.** Find $\mathbf{A}^{-1}$ using the cofactor representation. (Theorem 2 pg.318)

3.3. **Solutions to Linear Systems.** Using $\mathbf{A}^{-1}$ find the unique solution to $\mathbf{Ax} = \mathbf{b} = [b_1\, b_2\, b_3]^{\mathrm{T}}$.

4. Determinants

Given,

$$\mathbf{A} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

4.1. ***Vandermonde Determinant.*** Show that the $\det(\mathbf{A}) = (c-a)(c-b)(b-a)$.

4.2. **Application.** Determine which of the following sets of points can be uniquely interpolated by the polynomial $p(t) = a_0 + a_1 t + a_2 t^2$.

$$S_1 = \{(1,12),(2,15),(3,16)\}$$
$$S_2 = \{(1,12),(1,15),(3,16)\}$$
$$S_3 = \{(1,12),(2,15),(2,15)\}$$

5. Rotation Transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$

Given,

$$\mathbf{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

5.1. **The Unit Circle.** Show that the transformation $\mathbf{A}\hat{\mathbf{i}}$ rotates $\hat{\mathbf{i}} = [1\ 0]^{\mathrm{T}}$ counter-clockwise by an angle θ and defines a parametrization of the *unit circle*. What matrix would undo this transformation?

5.2. **Determinant.** Show that $\det(\mathbf{A}) = 1$.

5.3. **Orthogonality.** Show that $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$.

5.4. **Rotations in $\mathbb{R}^3$.** Given,

$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad \mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Describe the transformations defined by each of these matrices on vectors in $\mathbb{R}^3$.